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PLANE SYMMETRIC STRING COSMOLOGICAL MODEL WITH ZERO MASS SCALAR FIELD IN f(R) GRAVITY

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In this article, an anisotropic Locally Rotationally Symmetric (LRS) Bianchi type I metric in the presence of cloud string fluid and zero mass scalar field in reference to f *(*R*) gravity have been investigated. To obtain the deterministic solutions we assumed the weak field limit for a point-like source* $f(R)=R^{3/2}$ *and the very well-known expansion-shear scalar proportionality relation. Furthermore, some physical and kinematical parameters have been calculated to study the astrophysical consequences of obtained model, which shows a good resemblance to the recent observational data.*
Keywords: *cosmic string: scalar field: Bianchi type I metric: f (R) gravity*

1. Introduction

Observational evidence and measurements from high redshift supernovae observed by cosmologists [1-3] at various redshift ranges suggest that the universe is in its accelerating phase, and this acceleration is assumed to be the effect of dark energy, which is due to the universe's negative pressure. Two solutions have been brought forth to address this problem; one is to develop a viable dark energy model, and the other is to modify Einstein's theory of gravity. Among these non-Einsteinium theories, one of the theories is $f(R)$ gravity, this modification is one of

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the oldest and was originally proposed by Buchdahl [4]. Modified theories of gravitation received growing attention lately but the scientific curiosity of cosmologists for the universe gave them new inspiration to study the universe hugely in new ways. Despite the many shortcomings conferred to modified gravities, various cosmologist gave their tremendous efforts. Nojiri and Odintsov, [5] reconstructed cosmological methods i.e. inverse problems in view of cosmic time or e-folding in $f(R)$ gravity. Katore and Hatkar, [6] examined interacting as well as non-interacting scenarios of two fluids considering FRW space-time in $f(R)$ theory of gravity. Bahamonde et al., [7] studied modified teleparallel gravitational theories and derived the particular unification case of teleparallel equivalent to $f(R)$ gravitational theory which is invariant under local Lorentz transformation. Ferraro, [8] concisely reviewed the $f(R)$, and $f(T)$ gravity theories and showed some remarkable applications to cosmology and cosmic strings. De La Cruz-Dombriz and Dobado, [9] considered the possibility of describing the current evolution of the universe, without the introduction of any cosmological constant or dark energy, by modifying the Einstein-Hilbert action. Nojiri and Odintsov, [10] suggested the new model of modified gravity which contains positive and negative powers of the curvature in which the Lagrangian appears to be $L = R + R^m + 1/R^n$ where m and n are positive numbers. Again Nojiri and Odintsov, [11] suggested two realistic $f(R)$ and one $f(G)$ modified gravities which are consistent with local tests and cosmological bounds. Guendelman and Herrera, [12] analyzed unification: emergent universe followed by inflation and dark epochs from multi-field theory. Capozziello et al., [13] considered the tree-level effective gravitational action of bosonic string theory coupled with the diliaton field. Bari and Bhattacharya, [14] presented the full treatment of scalar and vector cosmological perturbations in a non-singular bouncing universe in the context of metric $f(R)$ cosmology. Sadeghi et al., [15] studied some cosmological parameters in a logarithmic corrected $f(R)$ gravitational model with swampland conjectures.

The investigation of yet unsolved interacting fields in reference to modified gravitation theories assuming one of the fields is a massless scalar field is a basic attempt to study the unification of the quantum and gravitational theories. In recent years, there has been a lot of interest in the set of field equations that represent a zero-mass scalar field coupled with gravitational theories. Venkateswarlu et al. [16-18], Godani and Samanta [19], Singh et al. [20], Singh [21], Patra [22], Adhav et al. [23], Dixit et al. [24], Katore et al. [25], Cadoni and Franzin [26], Pawar et al., [27], are some of the authors who have vigorously studied interacting fluid with one matter content as a zero mass scalar field.

In accordance with the study of strings, these are widely receiving significant interest from researchers as they play an important role in explaining the early phase of cosmic evolution. Nojiri et al., [28] studied string-inspired models, inflation, bounce, and late-time evolution in reference to modified gravity. Freidel et al., [29] discussed the formulation and dynamics of string theory and looks for string solutions. Pawar et al. [30-32] have studied string-coupled cosmological models using Bianchi type V and VI_0 space-time in the context of teleparallel gravity. Mishra et al., [33] investigated the string cosmological model using spatially homogeneous and anisotropic Bianchi type V space-time. The viscous string cosmological model explaining the cosmic accelerated expansion has been investigated by Vinutha et al., [34]. Darabi et al., [35] obtained string cosmological solutions via Hojman symmetry using FRW line element. Chirde et al., [36] have studied the LRS Bianchi type I cosmological model having the source as perfect fluid and a string of clouds using three different $f(T)$ formalisms. In modern cosmology, the substantial theoretical development of string theory [37-47], has been done using different types of gravitation theories.

Motivated by the situations discussed above in this paper, we have considered anisotropic LRS Bianchi type I space-time to construct a string cosmological model coupled with zero mass scalar field within the context of $f(R)$ gravity. This paper is divided into several sections: Sec. 2 deals with $f(R)$ gravity formalism. In Sec. 3 considering Bianchi type I metric, we have obtained the corresponding field equations. In Sec. 4, we obtained the exact solution of highly non-linear field equations along with different physical and kinematical quantities and presented them with 3D graphs. Lastly, in Sec. 5, we have concluded the investigations.

2. $f(R)$ gravity formalism

The $f(R)$ theory of gravitation is a modification of the general theory of relativity. The action for $f(R)$ gravity is given by

$$S = \int \sqrt{-g}\left(f(R) + L_m\right) d^4x \,, \tag{1}$$

where $f(R)$ is a general function of Ricci Scalar R and L_m is the matter Lagrangian. It is worth mentioning that the standard Einstein-Hilbert action can be recovered when $f(R) = R$.

The corresponding field equations are obtained by varying the action with respect to the metric $g_{\mu\nu}$ as

$$F(R)R_{\mu\nu} - \frac{1}{2} f(R) g_{\mu\nu} - \nabla_\mu \nabla_\nu F(R) + g_{\mu\nu} \nabla^\mu \nabla_\mu F(R) = T_{\mu\nu} \,. \tag{2}$$

where $F(R) \equiv df(R)/dR$, ∇_μ denotes covariant differentiation, $T_{\mu\nu}$ is the standard matter energy-momentum tensor derived from the matter Lagrangian L_m.

3. Metric and field equations

We consider an anisotropic LRS Bianchi type I metric of the form as

$$\mathrm{d}\, ds^2 = dt^2 - A^2 dx^2 - B^2\left(dy^2 + dz^2\right), \tag{3}$$

where A and B are functions of cosmic time t only.

The energy-momentum tensor for a one-dimensional cosmic string coupled with a zero mass scalar field is given by

$$T_{\mu\nu} = \rho u_\mu u_\nu - \lambda x_\mu x_\nu + \left(\psi_{,\mu}\psi_{,\nu} - \frac{1}{2} g_{\mu\nu} \psi_{,m}\psi^{,m} \right), \tag{4}$$

where λ and ρ are respective the tension density and rest energy density of string cloud fluid. u_μ denotes four time-like velocity vectors and x_μ denotes a unit space-like vector which represents the anisotropic direction of cloud string and satisfies the conditions,

$$g^{\mu\nu} u_\mu u_\nu = -x_\mu x_\nu = 1, \quad u_\mu x_\mu = 0. \tag{5}$$

In the comoving coordinate system, components of the energy-momentum tensor from equation (4) are given by

$$T_1^1 = -\lambda + \frac{1}{2}\dot{\psi}^2, \quad T_2^2 = T_3^3 = \frac{1}{2}\dot{\psi}^2 \quad \text{and} \quad T_4^4 = \rho - \frac{1}{2}\dot{\psi}. \tag{6}$$

Taking consideration of (6), the field equations (2) for the metric (3) are obtained as

$$\ddot{F} + 2\left(\frac{\dot{B}}{B}\right)\dot{F} + \left(\frac{\ddot{A}}{A} + 2\frac{\dot{A}\dot{B}}{AB}\right)F - \frac{1}{2}f = -\lambda + \frac{1}{2}\dot{\psi}^2, \tag{7}$$

$$\ddot{F} + \left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B}\right)\dot{F} + \left(\frac{\ddot{B}}{B} + \frac{\dot{B}^2}{B^2} + \frac{\dot{A}\dot{B}}{AB}\right)F - \frac{1}{2}f = \frac{1}{2}\dot{\psi}^2, \tag{8}$$

$$\left(\frac{\dot{A}}{A} + 2\frac{\dot{B}}{B}\right)\dot{F} + \left(\frac{\ddot{A}}{A} + 2\frac{\ddot{B}}{B}\right)F - \frac{1}{2}f = \rho - \frac{1}{2}\dot{\psi}^2, \tag{9}$$

where the overhead dot (.) denotes the derivative with respect to cosmic time *t*.

Here we have three non-linear differential field equations with six unknowns, namely; *f*, *A*, *B*, ρ, λ and ψ. The solution of these unknowns is discussed in the next section.

Also, we define some kinematical space-time quantities of physical interest in cosmology, as follows:

The average scale factor *a* and the spatial volume *V* are respectively defined as

$$a = \sqrt[3]{AB^2} \quad \text{and} \quad V = a^3. \tag{10}$$

The volumetric expansion rate of the universe is described by the generalized mean Hubble's parameter *H* given by

$$H = \frac{1}{3}\sum_{i=1}^{3} H_i = \frac{1}{3}\left(H_1 + H_2 + H_3\right), \tag{11}$$

in which $H_1 = \dot{A}/A$, and $H_2 = H_3 = \dot{B}/B$ denotes the directional Hubble's parameters.

Using (10) and (11), we have obtained the expansion scalar Θ, the mean anisotropy parameter Δ, the shear scalar σ^2, and the deceleration parameter q respectively as

$$\Theta = \frac{\dot{A}}{A} + 2\frac{\dot{B}}{B} = 3H, \tag{12}$$

$$\Delta = \frac{1}{3}\sum_{i=1}^{3}\left(\frac{H_i - H}{H}\right)^2, \tag{13}$$

$$\sigma^2 = \frac{1}{2}\left(\sum_{i=1}^{3} H_i^2 - \Theta^2\right), \tag{14}$$

$$q = -\frac{a\ddot{a}}{\dot{a}^2} = -1 + \frac{d}{dt}\left(\frac{1}{H}\right). \tag{15}$$

4. Solution of field equations

To solve the nonlinear differential field equations (7)-(9) and obtain the exact solution, we consider the weak field limit for a point-like source of $f(R)$ gravity model as presented by Capozziello et al. [48] and given by

$$f(R) = R^{3/2}. \tag{16}$$

For the deterministic solutions, we consider the expansion scalar Θ is proportional to the shear scalar σ^2 which leads to the following analytic relation

$$A = B^{\zeta}, \tag{17}$$

where ζ is a constant.

Also, as described by Yadav [49] we consider the average scale factor in the form

$$a(t) = (nDt)^{1/n}, \tag{18}$$

where n is a non-negative constant and D is an arbitrary constant.

We obtained the metric coefficients A and B as

$$A = (nDt)^{3\zeta/n(\zeta+2)}, \quad B = (nDt)^{3/n(\zeta+2)}. \tag{19}$$

Substituting values of A and B from (19) in (3), we get

$$\mathrm{d}\, ds^2 = dt^2 - (nDt)^{6\zeta/n(\zeta+2)} dx^2 - (nDt)^{6/n(\zeta+2)} (dy^2 + dz^2). \tag{20}$$

The metric coefficients of the model obtained in (20) are constant for any type of t, and hence it is free from any type of singularity.

In the following, we have determined the spatial volume V, the mean Hubble's parameter H, the expansion scalar Θ, the mean anisotropy parameter Δ, the shear scalar σ^2, and the deceleration parameter q respectively as

$$V = (nDt)^{3/n}, \tag{21}$$

$$H = \frac{1}{nt}, \tag{22}$$

$$\Theta = \frac{3}{nt}, \tag{23}$$

$$\Delta = \frac{2(\zeta - 1)^2}{(\zeta + 2)^2}, \tag{24}$$

where $\zeta \neq 1$, and $\zeta \neq -2$

$$\sigma^2 = \frac{3(\zeta - 1)^2}{n^2 t^2 (\zeta + 2)^2}, \tag{25}$$

where $\zeta \neq 1$, and $\zeta \neq -2$

$$q = n - 1. \tag{26}$$

The ratio

$$\frac{\sigma^2}{\Theta^2} = \frac{1}{3}\frac{(\zeta - 1)^2}{(\zeta + 2)^2}. \tag{27}$$

where $\zeta \neq 1$, and $\zeta \neq -2$.

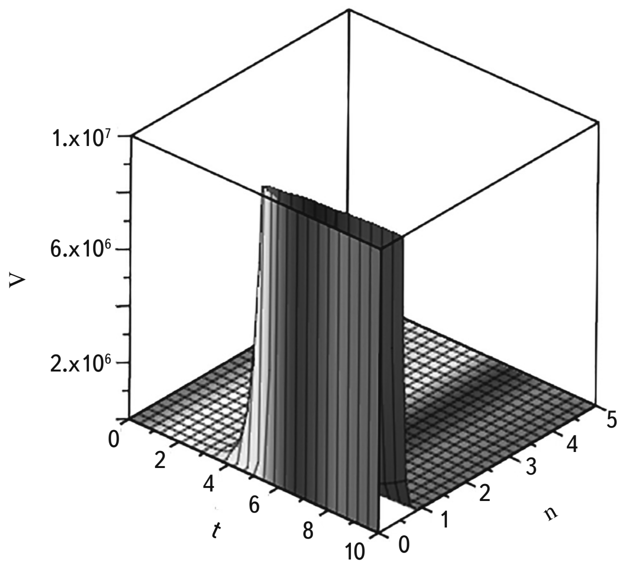

Fig. 1. Plot of V vs t for $\zeta = -0.875$ and $D = 3$.

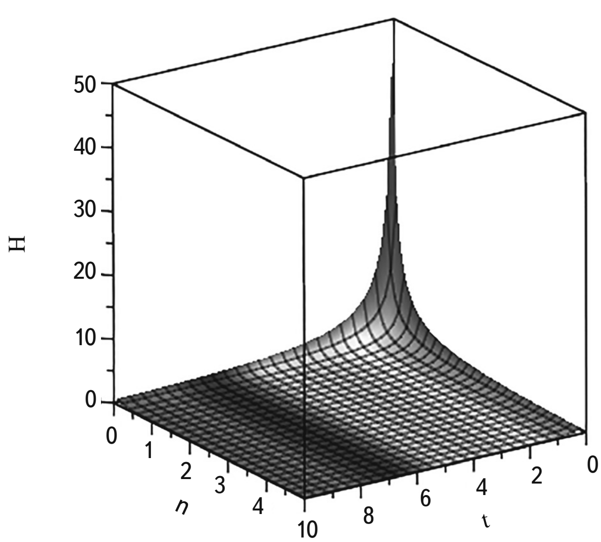

Fig. 2. Plot of H vs t.

It is observed from Fig.1 that the spatial volume is zero at the beginning of time and increases asymptotically with an increase in cosmic time for the range $0 < n < 1$ showing the expansion of the universe but for $n > 1$ the volume is found to be stable and flat, while the Hubble's parameter which is an inverse function of cosmic time decreases monotonically (Fig.2) with both the increase in cosmic time t and varying constant n and hence the rate of expansion of the universe also decreases. The ratio $\sigma^2/\Theta^2 \neq 0$ shows the discussed model doesn't approach isotropy. Also, the sign of q determines the accelerating or decelerating phase of the universe. The positive sign of q i.e. for $n > 1$ corresponds to a plain decelerating cosmological model although the deceleration parameter in range $-1 \leq q < 0$ corresponds to an accelerating universe and for $q = 0$ i.e. for $n = 1$ corresponds to the evolution with a constant rate. The observational SN Ia data [1,2] supports cosmic acceleration i.e $-1 \leq q < 0$.

The scalar field

$$\psi = -6t\left\{-\frac{2[(2n+3)\zeta + 4n][(n-2)\zeta + 2n-1][(n-3)\zeta^2 + 2(2n-3)\zeta + 4n-9]}{n^3(\zeta+2)^3 t^3[-6(n-3)\zeta^2 - 12(2n-3)\zeta - 6(4n-9)]^{1/2}}\right\}^{1/2}. \quad (28)$$

The behaviour of the scalar field observed in Fig.3 determines that the value of ψ that is negative throughout the evolution and is an asymptotically increasing function of cosmic time t.

The energy density

$$\rho = -\frac{9[(2n^2-5n-3)\zeta^2 + (8n^2-20n-15)\zeta + 4n(2n-5)][(n-3)\zeta^2 + 2(2n-3)\zeta + 4n-9]}{n^3(\zeta+2)^3 t^3[-6(n-3)\zeta^2 - 12(2n-3)\zeta - 6(4n-9)]^{1/2}}. \quad (29)$$

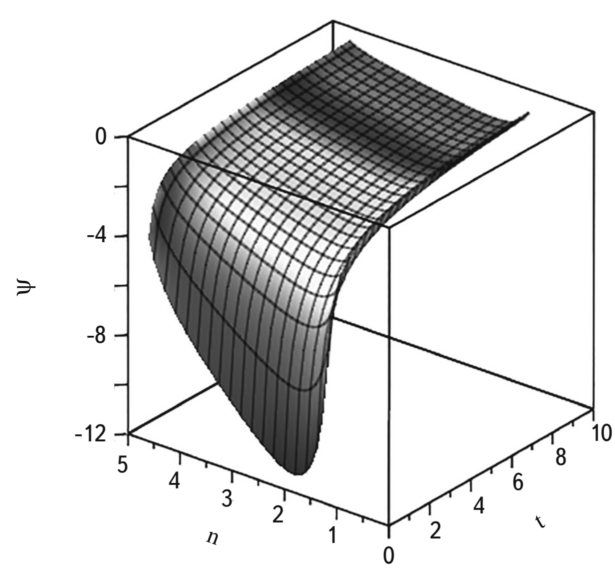

Fig. 3. Plot of ψ vs t for $\zeta = -0.875$.

The string tension density

$$\lambda = \frac{81(\zeta-1)\left[(n-3)\zeta^2 + 2(2n-3)\zeta + 4n-9\right]}{n^3(\zeta+2)^3 t^3\left[-6(n-3)\zeta^2 - 12(2n-3)\zeta - 6(4n-9)\right]^{1/2}}. \tag{30}$$

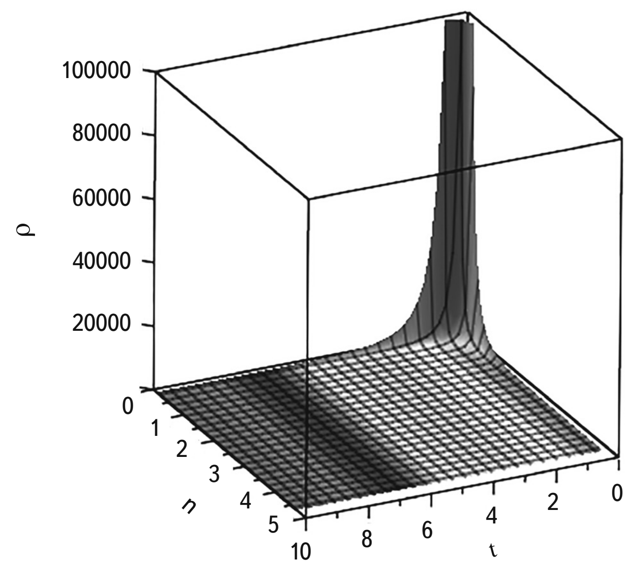

Fig.4. Plot of ρ vs t for $\zeta = -0.875$.

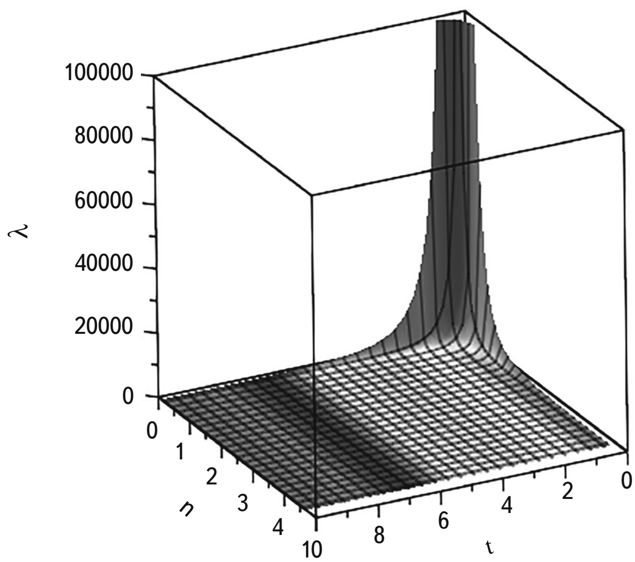

Fig.5. Plot of λ vs t for $\zeta = -0.875$.

It is observed from Fig.4 as well as from Fig.5 that as the cosmic time t and the value of constant n increases the energy density and the tension density both decreases to approach zero when $t \to \infty$. Furthermore, we observed that the presence of string as compared to the energy particles is quite larger which indicates the string dominance over energy particles. Letelier [44] studied the possibility that during the evolution of the universe, the strings disappear leaving only particles, and pointed out that the string tension density λ can be positive or negative. In our model, $\lambda > 0$ throughout the evolution, not only shows the presence of strings in the universe but also the string dominance over particles. Additionally, as both densities are positive, decrease with increasing cosmic time, and both approach zero when $t \to \infty$ indicates that the universe is expanding and expansion will keep forever which is in good agreement with [50].

The equation of state (EoS) for string fluid is given by

$$\rho = \varepsilon\lambda. \tag{31}$$

Then the EoS parameter can be obtained as

$$\varepsilon = -\frac{\left(2n^2 - 5n - 3\right)\zeta^2 + \left(8n^2 - 20n - 15\right)\zeta + 4n\left(2n - 5\right)}{9\left(\zeta - 1\right)\left(\zeta + 2\right)}. \tag{32}$$

The EoS parameter is observed to be a constant function and hence evolves constantly with increasing cosmic time. However, it should be noted that for $n \le 1$ the value of the EoS parameter decreases while with increasing $n > 1$ the value of the EoS parameter increases positively (Fig.6).

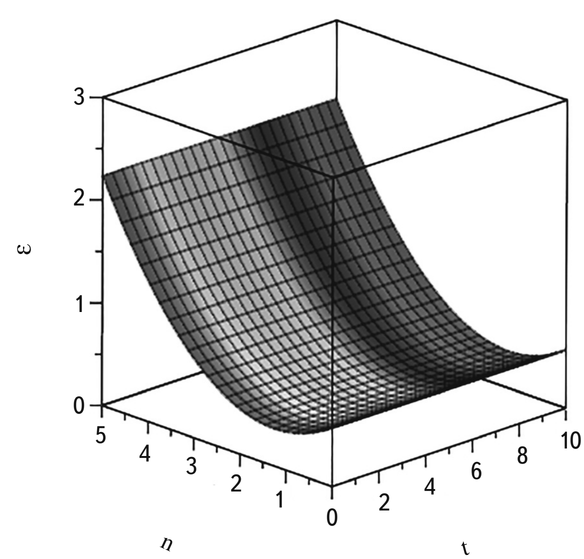

Fig.6. Plot of ε vs t for $\zeta = -0.875$.

The overall density parameter

$$\Omega=-\frac{3\left[\left(2n^2-5n-3\right)\zeta^2+\left(8n^2-20n-15\right)\zeta+4n(2n-5)\right]\left[(n-3)\zeta^2+2(2n-3)\zeta+4n-9\right]}{n(\zeta+2)^3\,t\left[-6(n-3)\zeta^2-12(2n-3)\zeta-6(4n-9)\right]^{1/2}}. \tag{33}$$

The graphical representation of the overall density parameter has been depicted in Fig.7 showing the complete positive epoch of the density parameter. It is observed that Ω decreases with an increase in cosmic time t and the value of constant n expressing the status for a flat universe which is supported by WMAP observations.

5. Concluding remarks

Zero mass scalar field and a string of clouds plays an vital role in understanding the early stages of cosmic evolution. In this present work, we have studied an anisotropic LRS Bianchi type I line element in the presence of one-dimensional cosmic string coupled with zero mass scalar field in reference to $f(R)$ gravity. The exact solutions to the field equations have been obtained by using the weak field limit for a point-like source of $f(R)$ gravity as presented by Capozziello et al. [48] and the average scale factor as described by Yadav [49]. It is observed that the constructed model is free from any type of singularity, expanding and showing acceleration or deceleration depending on the special choice of constant n which is in good agreement with recent observational data. The zero-mass scalar field is found to be negative which grows asymptotically in accordance with cosmic time t. The energy density and the tension density have finite values in the beginning which vanishes with an increasing cosmic time. Additionally, the obtained universe shows the string dominance over energy particles. Furthermore, the constructed model is flat

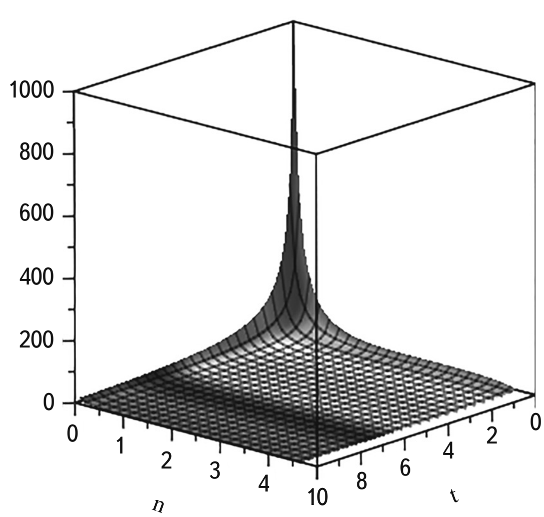

Fig.7. Plot of Ω vs t for $\zeta=-0.875$.

which is supported by WMAP observations, having constant EoS parameter and showing the never-ending expansion which is in good agreement with [50].

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