Classroom

In this section of *Resonance*, we invite readers to pose questions likely to be raised in a classroom situation. We may suggest strategies for dealing with them, or invite responses, or both. "Classroom" is equally a forum for raising broader issues and sharing personal experiences and viewpoints on matters related to teaching and learning science.

Exploring Kepler Problem Using Maple*

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The Kepler problem is named after Johannes Kepler, who proposed Kepler's laws of planetary motion. Kepler problem is a special case of the two-body problem in classical mechanics. The two bodies interact by a central force F that varies in strength as the inverse square of the distance r between them. Here, we are looking for the equation of motion using the Lagrangian formulation and its solution using a numerical approach. Though the general theory of relativity provides more accurate solutions to the two-body problem, especially in strong gravitational fields, here we are exploring numerical methods followed by correction term in potential energy, inversely proportional to the cube of the radius for perihelion motion.

1. Introduction

Keywords
Kepler problem, perihelion motion, Maple.

The Kepler problem refers to a classical problem in celestial mechanics that involves understanding the motion of two bodies in space under the influence of their mutual gravitational attraction. Specifically, it deals with studying the orbits of planets or other

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celestial objects around a central body, such as the motion of a planet around the Sun.

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The problem is named after the German astronomer Johannes Kepler, who formulated three fundamental laws of planetary motion in the early 17th century. These laws describe how planets orbit the Sun in elliptical paths, with the Sun at one focus of the ellipse. The laws are:

> **Kepler's First Law** (*Law of Ellipses*): Each planet orbits the Sun in an ellipse, with the Sun at one of the two foci of the ellipse.

Kepler's Second Law (Law of Equal Areas): A line segment joining a planet and the Sun sweeps out equal areas during equal intervals of time. This means that a planet moves faster when it is closer to the Sun and slower when it is farther from the Sun.

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> **Kepler's Third Law** (*Law of Harmonies*): The square of the orbital period of a planet (the time it takes to complete one orbit) is proportional to the cube of the semi-major axis of its orbit. This can be expressed as $T^2 \propto a^3$; here, T is the orbital period, and a a is the semi-major axis of the ellipse.

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The Kepler problem can be solved using the equations derived from these laws, and it has significant implications for our understanding of planetary motion and gravitational forces.

In a broader context, the Kepler problem is often used to refer to the general problem of determining the motion of two bodies under the influence of their mutual gravitational attraction. This problem is a specific case of the more general two-body problem in celestial mechanics, which involves solving for the orbits of two bodies orbiting their common center of mass.

The most important central-force problem is one involving a force proportional to the inverse square of the distance—the Kepler problem. Newton's law of gravitation is

$$\mathbf{F} = G\frac{mM}{r^2}\hat{r}.$$

If the potential energy V only depends on the mutual separation r of the two bodies, we describe such a force as a central force (because the force is always along r). It is convenient to handle central force problems in polar coordinates. One can prove that the two bodies remain in the same plane. Let the plane be $\theta = \pi/2$; the kinetic energy and potential energy in polar coordinates is

$$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2), \qquad V(r) = -\frac{GMm}{r}. \tag{1}$$

The Lagrangian is then

$$L = T - V(r). \tag{2}$$

This Lagrangian has no explicit dependence on ϕ. Because of the symmetry property, the corresponding conjugate momentum, namely angular momentum, is a conserved quantity of the system:

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} \equiv l = \text{constant}.$$

If a Lagrangian has no explicit dependence on time, the energy of the system is conserved:

$$\frac{\partial L}{\partial t} = 0 \Rightarrow T + V = \text{constant}.$$

Conserved quantities in mechanics, such as angular momentum and energy, are direct consequences of symmetries of the Lagrangian.

2. Kepler Problem

The Lagrangian [1] in central-force problem is

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) - \frac{GMm}{r}. \tag{3}$$

The angular momentum l bears relation in ϕ for $\dot{\phi} = \mathrm{d}\phi/\mathrm{d}t = \omega$ as:

$$mr^2\dot{\phi} = l. \tag{4}$$

2.1 *Formulation of Lagrange's Equation*

The Euler–Lagrange equations [2], or Lagrange's equations of the second kind

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j}. \tag{5}$$

Substituting following two equations in the Lagrange's formulation (5):

$$\frac{\partial L}{\partial \dot{r}} = m\dot{r} \Rightarrow \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{r}}\right) = m\ddot{r}$$

$$\frac{\partial L}{\partial r} = mr\dot{\phi}^2 - \frac{GMm}{r^2}.$$

Thus, we get Lagrangian for Kepler:

$$m\ddot{r} - mr\dot{\phi}^2 + \frac{GMm}{r^2} = 0. \tag{6}$$

2.2 *Solving Lagrange's Equation*

We have two differential equations—(4) (first-ordered) and (6) (second-ordered)—which we intend to solve numerically by exploring initial conditions, that is we need r_0, $\dot{r}_0$, ϕ_0, $\dot{\phi}_0$ arbitrarily (in fact meaningfully) as initial condition. It is generally convenient to set zero time at the moment that $\dot{r} = 0$ and to define $\phi_0 = 0$.

To make the equations of motion easier to analyse, we decouple these two equations to obtain equations of r and ϕ separately:

$$\dot{\phi} = \frac{l}{mr^2}. \tag{7}$$

Substituting (7) into (6) and simplifying, we get

$$m\ddot{r} - \frac{l}{mr^3} + \frac{GMm}{r^2} = 0. \tag{8}$$

In our approach to the Kepler problem [3], we made no use of the fact that energy is conserved; we simply obtained equations of motion from the Lagrangian and directly solved those differential equations (numerically). Energy conservation is a consequence of a symmetry property of the Lagrangian—its time independence. We exploit symmetry to simplify calculations so that a second-order differential equation is reduced to a first-order one. If we treat a mechanical problem as a problem of the calculus of variations, no knowledge about the energy is necessary; to find particle motion, we only need initial conditions.

Energy conservation is a consequence of a symmetry property of the Lagrangian—its time independence.

Energy and angular momentum are, nevertheless, related to initial conditions. Because both energy and angular momentum are conserved, their values calculated from time zero remain invariant:

$$E = \frac{1}{2}m(\dot{r}_0^2 + r_0^2\dot{\phi}_0^2) + V(r_0), \qquad l = mr_0^2\dot{\phi}_0^2. \tag{9}$$

Though Kepler problem admits an analytic solution, here, we envisage a numerical approach to solve the same using the property of conservation of energy and angular momentum:

$$E = T + V = \text{constant}, \qquad l = \text{constant}. \tag{10}$$

$$E = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) - \frac{GMm}{r}, \qquad l = mr^2\dot{\phi}^2. \tag{11}$$

Both the conservation equations (11) are first-ordered differential equation and can be combined to eliminate $\dot{\phi}$ to write a single first-ordered differential equation as:

$$E = \frac{1}{2}m\dot{r}^2 + \frac{l^2}{2mr^2} - \frac{GMm}{r}.$$

Above relation can be further rearranged, simplified for $\dot{r} = \mathrm{d}r/\mathrm{d}t$ in order to solve first ordered differential equation in variable separation mode as following:

$$\mathrm{d}t = \frac{\mathrm{d}r}{\sqrt{\frac{2}{m}\left(E + \frac{GMm}{r}\right) - \frac{l^2}{m^2r^2}}}. \tag{12}$$

Similarly, the second in equation (11) can be further rearranged, simplified for$\dot{\phi} = \mathrm{d}\phi/\mathrm{d}t$ to solve first ordered differential equation in variable separation mode as following:

$$\mathrm{d}t = \frac{mr^2}{l}\mathrm{d}\phi. \tag{13}$$

Table 1. Trajectory dependence on eccentricity ϵ. In 5th entry for perihelion motion, $\dot{\phi} = 1.3$, $\phi = r = \dot{r} = 0$ for $G = M = m = 1$.

Sl. no.	ϵ	Energy	Trajectory
1	$\epsilon = 0$	$E = \frac{1}{2}\frac{m(GMm)^2}{l^2}$	Circle
2	$\epsilon < 1$	$E < 0$	Ellipse
3	$\epsilon > 1$	$E > 1$	Hyperbola
4	$\epsilon = 1$	$E = 0$	Parabola
5	$\epsilon = 0.693$	-0.175	Perihelion

Combining last two equations (12) and (13) to eliminate time and write equation of trajectory as:

$$\frac{mr^2}{l}\mathrm{d}\phi = \frac{\mathrm{d}r}{\sqrt{\frac{2}{m}\left(E + \frac{GMm}{r}\right) - \frac{l^2}{m^2r^2}}}.$$

Rearranging and integrating last equation, we get:

$$\phi = \int \frac{l}{r^2\sqrt{2m\left(E + \frac{GMm}{r}\right) - \frac{l^2}{r^2}}}\mathrm{d}r.$$

On solving this integration using Maple or by other method, we get

$$\phi = \tan^{-1}\left(\frac{m^2rGM - l^2}{l\sqrt{2mr^2E + 2m^2rGM - l^2}}\right) + \text{constant}. \tag{14}$$

2.3 *Polar Form of Solution*

Carrying out trigonometric rearrangements in the equation of conic section in polar coordinates, we get:

$$r = \frac{l^2}{m(GMm)}\frac{1}{(1 + \epsilon\cos\theta)}. \tag{15}$$

Here, ϵ represents eccentricity and is expressed as:

$$\epsilon = \sqrt{1 + \frac{2l^2E}{m(GMm)^2}}. \tag{16}$$

The shape of the orbit depends upon the eccentricity (ϵ) and *Table* 1 depicts its values for our geometric interests.

l	ϵ	Energy	Trajectory
1.0	0.0	0.5	Circle
1.2	0.44	−0.28	Ellipse
1.5	1.25	0.125	Hyperbola
1.414	1.0	0.0	Parabola
1.3	0.693	−0.175	Perihelion

Table 2. Trajectory deciding initial condition $l = \dot{\phi}$ and calculated quantities ϵ and E at a glance used in generating Maple plots.

3. Exploring Maple

The entire problem can be explored in Maple and then the trajectory equations further used to generate polar plots.

3.1 *Solving the Integration*

The solution of the above integration represented as equation (14) can be found in Maple with the following code:

```
Epr1 := 1/(r^2*sqrt(2*m*(En+G*M*m/r)-(l/r)^2));
Epr2 := int(Epr1, r);
Epr3 := simplify(Epr2);
```

$$\sqrt{2}\arctan\left(\sqrt{r-1}\right)$$

3.2 *Solving the DE and Plotting Various Trajectories*

In Maple, we are raising the expressions for kinetic and potential energy, Lagrangian and Lagrangian formulation, numerically solving and plotting them. The following code may be executed to maneuver various trajectories. In this process, initial boundary conditions play a vital role. *Table* 2 depicts the choice of $\dot{\phi}$ that decides ϵ and energy E.

```
restart;
T := (1/2)*m*((diff(r(t), t))^2+r(t)^2*(diff(phi(t), t))^2);
V := -G*M*m/r(t);
L := T-V;
L1 := subs({diff(phi(t), t) = var4, diff(r(t), t) =
```

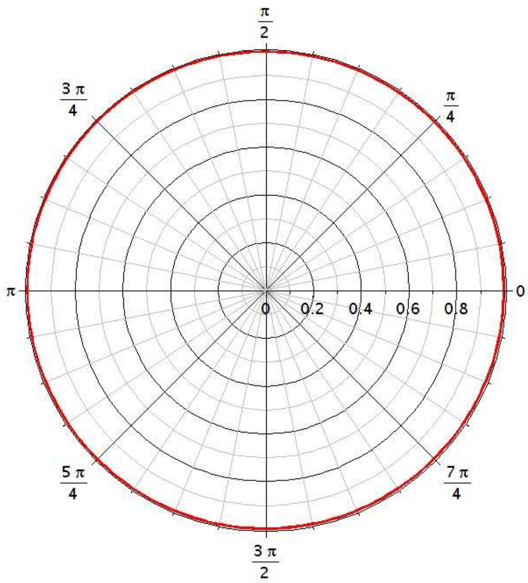

Figure 1. Circular trajectory for initial condition $\dot{\phi} = 1$, $\epsilon = 0$, $E = 0.5$.

```
var2, phi(t) = var3, r(t) = var1}, L);
Epr11 := diff(L1, var4);
Epr12 := diff(L1, var3);
Epr13 := subs({var1 = r(t), var2 = diff(r(t), t),
var3 = phi(t), var4 = diff(phi(t), t)}, Epr11);
Eq14 := Epr13 = l;
Epr21 := diff(L1, var2);
Epr22 := diff(L1, var1);
Epr23 := subs({var1 = r(t), var2 = diff(r(t), t),
var3 = phi(t), var4 = diff(phi(t), t)}, Epr21);
Epr24 := subs({var1 = r(t), var2 = diff(r(t), t),
var3 = phi(t), var4 = diff(phi(t), t)}, Epr22);
Epr25 := diff(Epr23, t);
Eq26 := Epr25-Epr24 = 0;
```

$$m\frac{d^2}{dt^2}r(t) - mr(t)\left(\frac{d}{dt}\phi(t)\right)^2 + \frac{GMm}{(r(t))^2} = 0$$

3.3 *The Circular Trajectory*

Introducing the initial condition $l = \dot{\phi} = 1$:

```
Eq31 := isolate(Eq14, diff(phi(t), t));
```

```
Eq32 := eval(Eq26, Eq31);
with(plots);
G := 1; M := 1; m := 1;
Eq41 := r(0) = 1;
Eq42 := (D(r))(0) = 0;
Eq43 := phi(0) = 0;
Eq44 := (D(phi))(0) = 1;
En := eval(T+V, {diff(phi(t), t) = rhs(Eq44),
diff(r(t), t) = rhs(Eq42), r(t) = rhs(Eq41)});
```

$\frac{1}{2}$

```
l := eval(lhs(Eq14), {diff(phi(t), t)
= rhs(Eq44), r(t) = rhs(Eq41)})
```

1

```
\epsilon:= sqrt((2*En*l^2+1)/m(G*M*m)^2)
```

0

```
ini1 := Eq41, Eq42, Eq43;
Eq51 := dsolve({Eq31, Eq32, ini1}, {phi(t),
r(t)}, numeric, output = listprocedure);
polarplot([rhs(Eq51(t)[3]), rhs(Eq51(t)[2]),
t = -Pi .. Pi], scaling = constrained,
thickness = 4, color = red, axesfont =
["HELVETICA", "ROMAN", 14]);
```

$\mathrm{D}(\phi)(0) = 1.2$

3.4 *The Elliptical Trajectory*

Introducing the initial condition $l = \dot{\phi} = 1.2$:

```
Eq64 := (D(phi))(0) = 1.2:
En := eval(T+V, {diff(phi(t), t) = rhs(Eq64),
diff(r(t), t) = rhs(Eq42), r(t) = rhs(Eq41)})
```

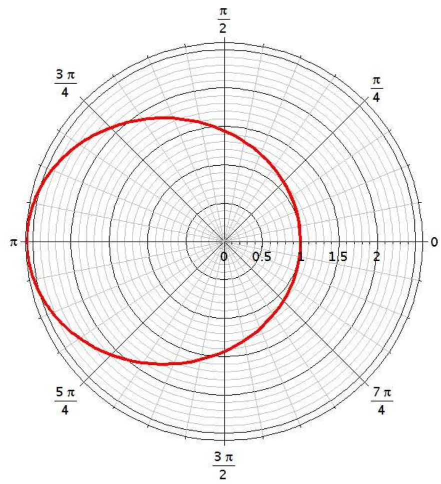

Figure 2. Elliptical trajectory for initial condition $\dot{\phi} = 1.2, \epsilon = 0.4, E = -0.28$.

−.2800000000

```
l := eval(lhs(Eq14), {diff(phi(t), t) = rhs(Eq64),
  r(t) = rhs(Eq41)});
```

1.2

```
\epsilon:= sqrt((2*En*l^2+1)/m(G*M*m)^2);
```

0.4400000000

```
ini2 := Eq41, Eq42, Eq43:
Eq71 := dsolve({Eq31, Eq32, ini2}, {phi(t),
r(t)}, numeric, output = listprocedure);
polarplot([rhs(Eq71(t)[3]), rhs(Eq71(t)[2]),
 t = 0 .. 5*Pi], scaling = constrained,
 thickness = 4, color = red, axesfont =
 ["HELVETICA", "ROMAN", 14]);
```

3.5 *The Hyperbolic Trajectory*

Introducing the initial condition $l = \dot{\phi} = 1.5$:

```
Eq84 := (D(phi))(0) = 1.5;
```

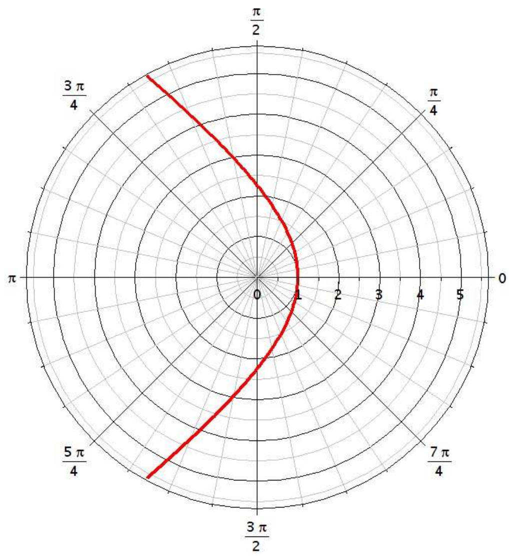

Figure 3. Hyperbolic trajectory for initial condition $\dot{\phi} = 1.5, \epsilon = 1.25, E = 0.125$.

```
En := eval(T+V, {diff(phi(t), t) =
rhs(Eq84), diff(r(t), t) = rhs(Eq42),
r(t) = rhs(Eq41)});
```

0.125000000

```
l := eval(lhs(Eq14), {diff(phi(t), t) =
rhs(Eq84), r(t) = rhs(Eq41)});
\epsilon:= sqrt((2*En*l^2+1)/m(G*M*m)^2);
```

1.250000000

```
ini2 := Eq41, Eq42, Eq43;
Eq91 := dsolve({Eq31, Eq32, ini2}, {phi(t),
r(t)}, numeric, output = listprocedure);
polarplot([rhs(Eq91(t)[3]), rhs(Eq91(t)[2]),
t = -2*Pi .. 2*Pi], scaling = constrained,
thickness = 4, color = red, axesfont =
["HELVETICA", "ROMAN", 14]);
```

3.6 *The Parabolic Trajectory*

Introducing the initial condition $l = \dot{\phi} = \sqrt{2} = 1.414$:

```
Eq104 := (D(phi))(0) = sqrt(2);
En := eval(T+V, {diff(phi(t), t) =
rhs(Eq104), diff(r(t), t) = rhs(Eq42),
r(t) = rhs(Eq41)});
```

0

```
l := eval(lhs(Eq14), {diff(phi(t), t) =
rhs(Eq104), r(t) = rhs(Eq41)})
```

$\sqrt{2}$

```
\epsilon:= sqrt((2*En*l^2+1)/m(G*M*m)^2)
```

1

```
ini2 := Eq41, Eq42, Eq43;
Eq111 := dsolve({Eq31, Eq32, ini2},
{phi(t), r(t)}, numeric, output = listprocedure);
polarplot([rhs(Eq111(t)[3]), rhs(Eq71(t)[2]),
t = -2*Pi .. 2*Pi], scaling = constrained,
thickness = 4, color = red, axesfont =
["HELVETICA", "ROMAN", 14]);
```

3.7 *Solving the DE and Plotting the Perihelion Trajectory*

Mercury is observed to move in an elliptical orbit, but this orbit does not quite close upon itself. The ellipse rotates, and the perihelion[1] advances; this phenomenon is called precession. In our consideration, planetary motion is governed purely by a potential of form $1/r$. Any departure from this idealized potential produces precession. Although the Sun is the dominant source of gravity in the solar system, other planets also exert an influence on Mercury. We can write the correction terms as expansion of r^{-1} to various powers. Furthermore, the inverse-square law of force applies to two spherical bodies. The Sun is not perfectly spherical because

[1] The point on its orbit nearest to the Sun.

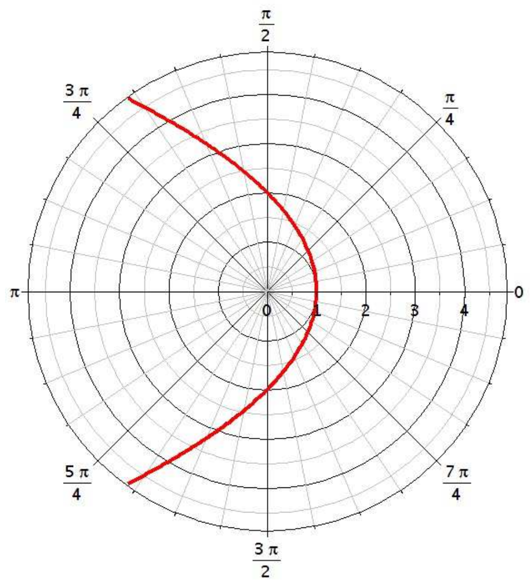

Figure 4. Parabolic trajectory for initial condition $\dot{\phi} = 1.414, \epsilon = 1.0, E = 0$.

the Sun rotates with a period of about 25 days, which causes a solar equatorial bulge. This oblate Sun can be considered to have a quadrupole moment, for which the potential energy is proportional to r^{-3}. According to our numerical treatment of the Kepler problem, we readily proceed to include this contribution to potential energy; with such a term in our Lagrangian, we also obtain a precessing orbit. Now, just adding the correction term to potential energy in equation (3) will cater to the purpose, followed by executing the rest of the Maple worksheet. The Lagrangian with the corrected term in potential energy will be:

$$L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\phi}^2\right) - \frac{GMm}{r} - k'\frac{1}{r^3}. \tag{17}$$

Here, we suppose $k' = 0.02$ and introduce the initial condition $l = \dot{\phi} = 1.3$ with our obvious scaling $G = M = m = 1$, follow the entire Maple code as depicted below and look for the perihelion trajectory plot.

```
restart;
T := (1/2)*m*((diff(r(t), t))^2+r(t)^2*
(diff(phi(t), t))^2);
```

```
V := -G*M*m/r(t)-0.2e-1/r(t)^3;
L := T-V;
L1 := subs({diff(phi(t), t) = var4, diff(r(t), t)
= var2, phi(t) = var3, r(t) = var1}, L);
Epr11 := diff(L1, var4);
Epr12 := diff(L1, var3);
Epr13 := subs({var1 = r(t), var2 = diff(r(t), t),
var3 = phi(t), var4 = diff(phi(t), t)}, Epr11);
Eq14 := Epr13 = l;
Epr21 := diff(L1, var2);
Epr22 := diff(L1, var1);
Epr23 := subs({var1 = r(t), var2 = diff(r(t), t),
var3 = phi(t), var4 = diff(phi(t), t)}, Epr21);
Epr24 := subs({var1 = r(t), var2 = diff(r(t), t),
var3 = phi(t), var4 = diff(phi(t), t)}, Epr22);
Epr25 := diff(Epr23, t);
Eq26 := Epr25-Epr24 = 0;
Eq31 := isolate(Eq14, diff(phi(t), t));
Eq32 := eval(Eq26, Eq31);
```

$$m\frac{d^2}{dt^2}r(t) - \frac{l^2}{m(r(t))^3} + \frac{GMm}{(r(t))^2} + 0.06\ (r(t))^{-4} = 0$$

```
with(plots);
G := 1; M := 1; m := 1;
Eq41 := r(0) = 1;
Eq42 := (D(r))(0) = 0;
Eq43 := phi(0) = 0;
Eq44 := (D(phi))(0) = 1.3;
En := eval(T+V, {diff(phi(t), t) = rhs(Eq44),
diff(r(t), t) = rhs(Eq42), r(t) = rhs(Eq41)});
```

$-.1750000000$

```
l := eval(lhs(Eq14), {diff(phi(t), t) =
rhs(Eq44), r(t) = rhs(Eq41)});
\epsilon;‘ := sqrt((2*En*l^2+1)/m(G*M*m)^2);
```

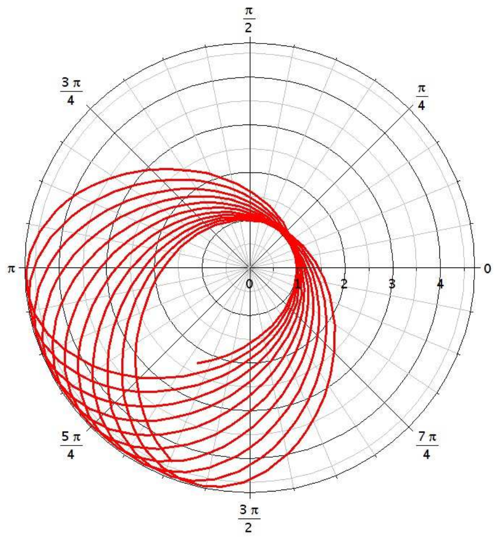

Figure 5. Perihelion motion for initial condition $\dot{\phi} = 1.3, \epsilon = 0.693, E = -0.175$.

0.6391400473

```
ini1 := Eq41, Eq42, Eq43;
Eq51 := dsolve({Eq31, Eq32, ini1}, {phi(t),
 r(t)}, numeric, output = listprocedure);
polarplot([rhs(Eq51(t)[3]), rhs(Eq51(t)[2]),
t = -Pi .. 100*Pi], scaling = constrained,
thickness = 3, color = red, axesfont =
["HELVETICA", "ROMAN", 14]);
```

4. Conclusions

The Kepler problem can be explored interactively, allowing for plotting various orbital trajectories, including the intriguing perihelion shift. Experimenting with eccentricity within the range $0 < \epsilon < 1$ provides insight into the elliptical paths and perihelion behavior. Starting with the Lagrangian of the central force in the Kepler problem, as given in equation (6), we can solve the first-order differential equation (11). Simplifying by eliminating the angular component makes it easier to apply the method of separation of variables for further analysis. The solution can be expressed in polar coordinates, which can be plotted using any programming language familiar to the reader. The authors have

Experimenting with eccentricity within the range $0 < \epsilon < 1$ provides insight into the elliptical paths and perihelion behavior.

provided examples in Maple, and even a C-language interface can be utilized to explore and fully understand the Kepler problem.

Suggested Reading

[1] H Goldstein, *Classical Mechanics*, 2nd Edn., Addison Wesley, 1980.

[2] I L Shapiro, *Lecture Notes on Newtonian Mechanics*, Springer, p.203, 2013.

[3] C Leivai, J Saavedra and J R Villanueva, The Kepler problem in the Snyder space, *Pramana: Journal of Physics*, Vol.80, No.6, pp.945—951, 2013.

[4] Refer complete programme on Github, https://github.com/swanwane/Kepler-Problem-in-MAPLE-2020

[5] Refer complete programme on Github, https://github.com/swanwane/Kepler-Problem/blob/main/The%20trajectories