Lorentz transformations using Complex Numbers

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ABSTRACT: Imaginary numbers are just not the part of mere mathematical jugglery; it carries relevance in the real world. Mathematical models involving imaginary time are efficient to predict effects that we observe. Here, we configure the established special relativity in complex numbers. Further, in quest of human race to develop a mathematical model that describe the universe we live in, we look further for handling effects which are never been observable.

KEYWORDS: Special relativity, Lorentz transformation, Complex numbers, Conformal mapping, Proper velocity

1. Introduction

The space-time event may be chart as $z = x + iy$ can also be thought as an ordered pair (x, y) in argand plane where we enforce obvious substitutions $x = ct$ $y = r$ to thus obtain $z = ct + ir$ and ordered pair (ct, r) while in another frame the quantities may be $z' = x' + iy'$ or $z = ct' + ir'$ abiding Einstein's special relativity. To properly scale our expressions we shall prefer velocity of light unity by considering $c = 1$ which also in turn offer simplicity again without the loss of generality. Thus, we consider $z = t + ir$ or ordered pair (t, r) and for another inertial frame $z' = t' + ir'$ or (t', r').

Now, the square of the complex number charting an event on simplification yields another complex number say w; technically - conformal mapping from $z \to w$ plane in which plot corresponds to function $w = f(z)$. In this way, a given function f assigns to each point in z in its domain of definition D the corresponding point $w = f(z)$ in w-plane, we say that f defines mapping into w-plane. The conformal mapping of complex plane

$$z^2 = (t^2 - r^2) + i\,2tr = S^2 + i\,W^2$$

Thus, we have $w = z^2$, thus possesses ordered pair $(S^2, W^2) \equiv (t^2 - r^2, 2rt)$. It is interesting to note here that the invariant quantity is special relativity is the '*interval*' which is denoted as by S can be picked up as $\sqrt{\operatorname{Re}\{z^2\}}$ or $\sqrt{\operatorname{Re}\{w\}}$ which corresponds to hyperbola $t^2 - r^2 = S^2$. Similarly imaginary part $\sqrt{\operatorname{Im}\{z^2\}}$ or $\sqrt{\operatorname{Im}\{w\}}$ corresponds again to hyperbola $W^2 = 2rt$. The later hyperbola is from the family of the former hyperbola obtained after $\frac{\pi}{4}$ rotation about *z*-axis with $S^2 = \frac{1}{2}W^2$.

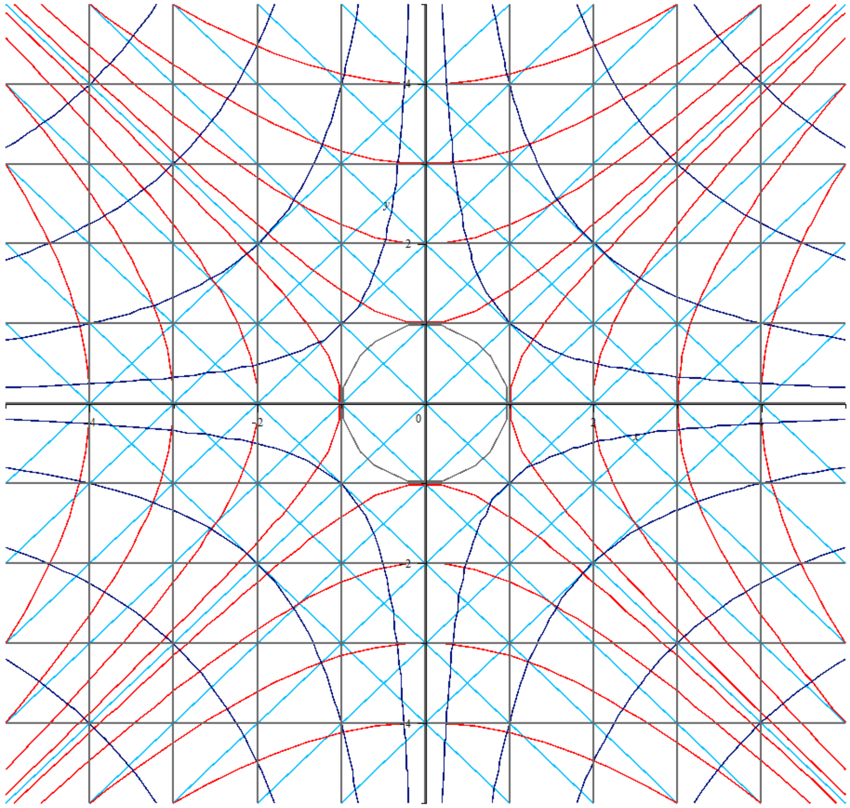

Figure 1. Grid of lines for real-part $\sqrt{t^2 - r^2} = S = 0, \pm 1, \pm 2, \ldots$ and imaginary-part $\sqrt{2xy} = W = 0, \pm 1, \pm 2, \ldots$ formed by hyperbolas with involved constants S and W which are universal for all inertial observers.

Here we propose modified Argand plane wherein we plot quantities S along real axis which are invariant quantities of special relativity and the beauty is that these are essentially the same for all inertial observers while quantities W on imaginary axis which sounds derived versions of S, hereafter referred as *SW-Argand* plane or simply *SWA* plane. To unfold the structure of *SWA* plane, we shall look at elemental grid equations; $\sqrt{t^2 - r^2} = S = 0, \pm 1, \pm 2, \ldots$, $\sqrt{2rt} = W = 0, \pm 1, \pm 2, \ldots$ and plots as depicted in Figure-1

In Euler representation the involved complex numbers follows as;

$$z = t + ir = Re^{i\varphi}, R = \sqrt{t^2 + r^2}, \varphi = \tan^{-1}\frac{r}{t} = \tan^{-1} v$$

$$w = z^2 = t^2 - r^2 + i2rt = R^2 e^{i2\varphi}, R^2 = t^2 + r^2, 2\varphi = \tan^{-1}\frac{2rt}{t^2 - r^2} = \tan^{-1}\frac{2v}{1 - v^2} = \tan^{-1}\left(2v\gamma^2\right)$$

As we represent the *event* in a complex form as z which depends upon time depicted on real axis as ct and space on imaginary axis depicted as r that is mathematically expressed as;

$$z(ct, r) = ct + i\, r$$

In case if the object is not moving, i.e. r is not function of time and its arbitrarily assigned value $r = r_0$ develops a world-line parallel to real axis at r_o.

In another case of an object moving with uniform velocity $v = \frac{d}{dt} r(t)$, we have; $r(t) = v\, t$ for appropriate boundary conditions

$$z(ct, r) = ct + i\, vt = \sqrt{(ct)^2 + (vt)^2}\, e^{i \tan^{-1} \frac{v}{c}} = ct\sqrt{1+\beta^2}\, e^{i \tan^{-1} \beta} = t\sqrt{1+v^2}\, e^{i \tan^{-1} v}$$

Thus, above equation represents a world-line inclined with real axis at an angle $\varphi = \tan^{-1} \frac{v}{c}$ and time ticked at the rate of c which is universal constant according to Einstein's Principle of Relativity (EPR). An inertial observer with reference to a frame F travelling with uniform velocity v at any time t possess location described by modulus $ct\sqrt{1+\beta^2}$ which depends linearly on time and its velocity β. The argument is the angle that the line OZ makes with real axis expressed as; $\varphi = \tan^{-1} \beta$. The slope of the line is obviously $\tan\varphi = \beta$. A faster moving object will have higher angle with real axis that limits up to $\pi/4$ as nothing can travel faster than light. However its location from origin of space-time coordinates floats away at rate $c\sqrt{1+\beta^2}$ per unit sec.

2. Complex Lorentz Transformation as Conformal Mapping

The *Lorentz transformation* equations with usual notations; $x' = \gamma(x - \beta y)$ and $y' = \gamma(y - \beta x)$ for obvious substitution $x = ct$ and $y = r$ we get; $ct' = \gamma(ct - \beta r)$ and $r' = \gamma(r - \beta ct)$ be expressed by a single elegant expression as;

$$z' = \gamma(z - i\beta\, \bar{z})$$

The above expression is known as a ***Complex Lorentz transformation*** equation which includes a pair of usual Lorentz Transformation equations where, $z' = x' + i\, y'$, $\bar{z} = ct - ir$ represents complex conjugate of $z = ct + ir$ and the reader may explore $\mathrm{Re}\{z'^2\} = \mathrm{Re}\{z^2\} \Rightarrow c^2t'^2 - r'^2 = c^2t^2 - r^2 = S^2$

If we consider the clock in moving frame of reference F' located at origin we opt mathematically $r' = 0$ and use a substitution $\tau = t'$. Thus, we get;

$$c^2\tau^2 = c^2t^2 - r^2 = S^2$$

$$\tau = t\sqrt{1-\left(\frac{r}{ct}\right)^2} = \frac{S}{c}$$

If we choose velocity $v = r/t$ and use γ for compact representation, its differential form appears as; $\frac{dt}{d\tau} = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \gamma$.

The ***complex velocity*** (ϑ) can just be obtained by time differentiation of the event equation $z = ct + i\,r$. Here, c is absolute constant may be considered as unity for scaling graphs;

$$\frac{dz}{dt} = \vartheta = c + i\frac{dr}{dt} = 1 + i\,v = \sqrt{1+v^2}\,e^{i\tan^{-1}v}$$

Treating $v = \frac{dr}{dt}$ as usual/ordinary definition of velocity that forms imaginary part and $c = 1$ forms real part to constitute the newly devised term as ***complex velocity*** ϑ. Note that, while representing the complex velocity, the argument in velocity-argand plane is $\sqrt{1+v^2}$ and the argument is ; $\varphi = \tan^{-1}v$. As the relative velocity v increases (*i*) the modulus or imaginary part in the argand plane also grows while the real part remains fixed at value preferred for scaling i.e. at $c = 1$ (*ii*) the argument also increases bound to $\pi/2$ for infinite velocity. This leads to the classical case wherein there is no customary to the assigned usual/ordinary velocity and that can attain infinitely large value. Note that $\vartheta^2 = (1-v^2) + i\,2v = (1+v^2)e^{i2\tan^{-1}v}$ and $\mathrm{Re}\{\vartheta^2\} = (1-v^2) = \gamma^{-2}$ is an invariant quantity which may be related to Einstein's Special Relativity.

The **proper complex velocity** can be devised by obtaining proper-time τ derivative of last equation treating v as uniform velocity (inertial frame) and c as absolute constant;

$$\dot{z} = \upsilon = c\gamma + iv\gamma = \gamma\sqrt{c^2+v^2}\,e^{i\tan^{-1}\frac{v}{c}}$$

Treating $\dot{z} = \frac{d}{d\tau}z = \upsilon$, $\dot{t} = \frac{d}{d\tau}t = \gamma$, $\dot{r} = \frac{d}{d\tau}r = \frac{dr}{dt}\frac{dt}{d\tau} = \frac{dr}{dt}\gamma = v\gamma$, where v is usual/ordinary velocity. Note that while representing the proper complex velocity, the argument in velocity-argand plane is $\gamma\sqrt{1+v^2} = \sqrt{\frac{1+v^2}{1-v^2}}$ and the argument is same as; $\varphi = \tan^{-1}v$. As the relative velocity β increases the modulus also grows and blows up at $v = c$.

To represent in a properly scaled manner we prefer $c=1$ hence, the equations takes form as;

$$\dot{z} = \upsilon = \gamma + iv\gamma = \sqrt{\frac{1+v^2}{1-v^2}} e^{i \tan^{-1} v}$$

It represents a point in the velocity-argand plane and the modulus behaves quite close to γ and blows at $v=c$ as depicted in the Figure 2. However, its argument roles from the 0 to $\pi/4$ for velocity ranging from 0 to 1 (i.e. $v=c=1$).

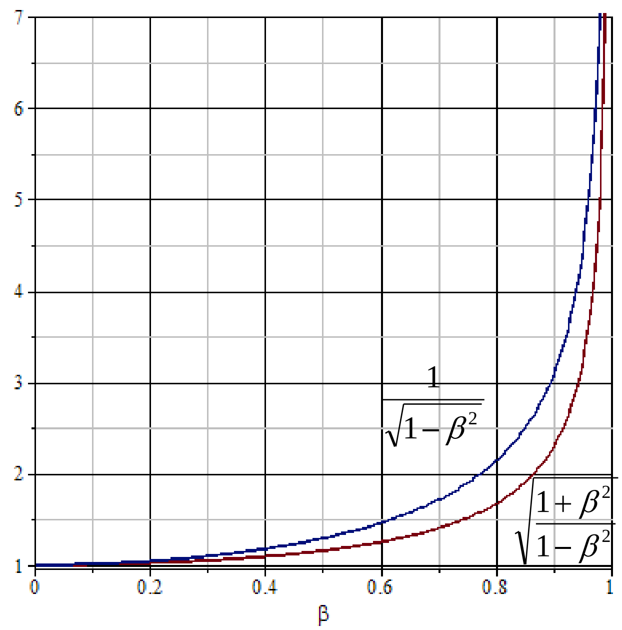

Figure 2 Variation of $\frac{1}{\sqrt{1-\beta^2}}$ and $\sqrt{\frac{1+\beta^2}{1-\beta^2}}$ with β.

The important aspect of generating a conformal map $w=\upsilon^2$ a invariant quantity is to find υ^2 and its real part filters out invariant quantity.

It's Conformal mapping $w=\upsilon^2=\gamma^2(1-v^2)+i(2\gamma^2 v)=1+i\frac{2v}{1-v^2}=U+iV$ here $U=1$ and $V=\frac{2v}{1-v^2}$.

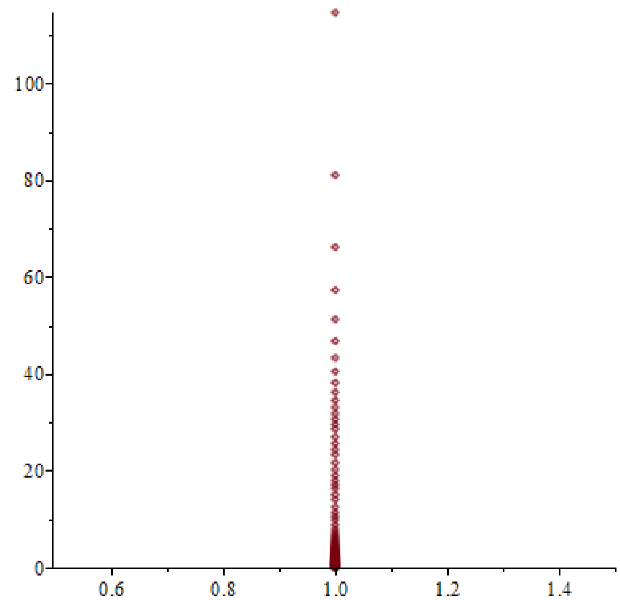

Figure 3 Argand plot for $w=\upsilon^2=1+i\frac{2v}{1-v^2}=U+iV$ as a parametric function of relative velocity v where real part $U=1$ remains constant while imaginary part $V=\frac{2v}{1-v^2}$ grows as a function of v.

Its Euler version may be represented as; $w=\upsilon^2=\frac{1+v^2}{1-v^2}e^{i2\tan^{-1}v}=R\,e^{i\phi}$

Here $U=1$ and $V=\frac{2v}{1-v^2}$ represents complex number in mapped w plane with U on real axis and V on imaginary axis. Following Figure 3 depicts that conformally maped complex function w blows as $v\to 1$.

The steady object is represented by $\upsilon=0+i\,1$ while the fastest, say $v=0.9999$ move with $\upsilon=70.70537+i\,70.71245$ units. The real part track the imaginary component as particle accelerated to approach speed of light.

Composite Velocity defines the way we add or subtract relative velocities in special relativity. Now, we shall express the velocities in a moving inertial frame F' as;

$$z'=ct'+ir'=\gamma(ct-\beta r)+i\gamma(r-\beta ct)$$

$$Mod(z')=\gamma\sqrt{(ct-\beta r)^2+(r-\beta ct)^2}=\gamma\sqrt{t^2+(\beta r)^2-2t\beta r+(r)^2+(\beta t)^2-2r\beta t}$$

$$Mod(z') = \gamma\sqrt{(1+\beta^2)(c^2t^2+r^2) - 4r\beta ct} = \sqrt{\frac{(1+\beta^2)(t^2+r^2) - 4r\beta t}{1-\beta^2}}$$

$$Arg(z') = \tan^{-1}\left(\frac{r-\beta ct}{ct-\beta r}\right) = \tan^{-1}\left(\frac{r/ct - \beta}{1-\beta\, r/ct}\right)$$

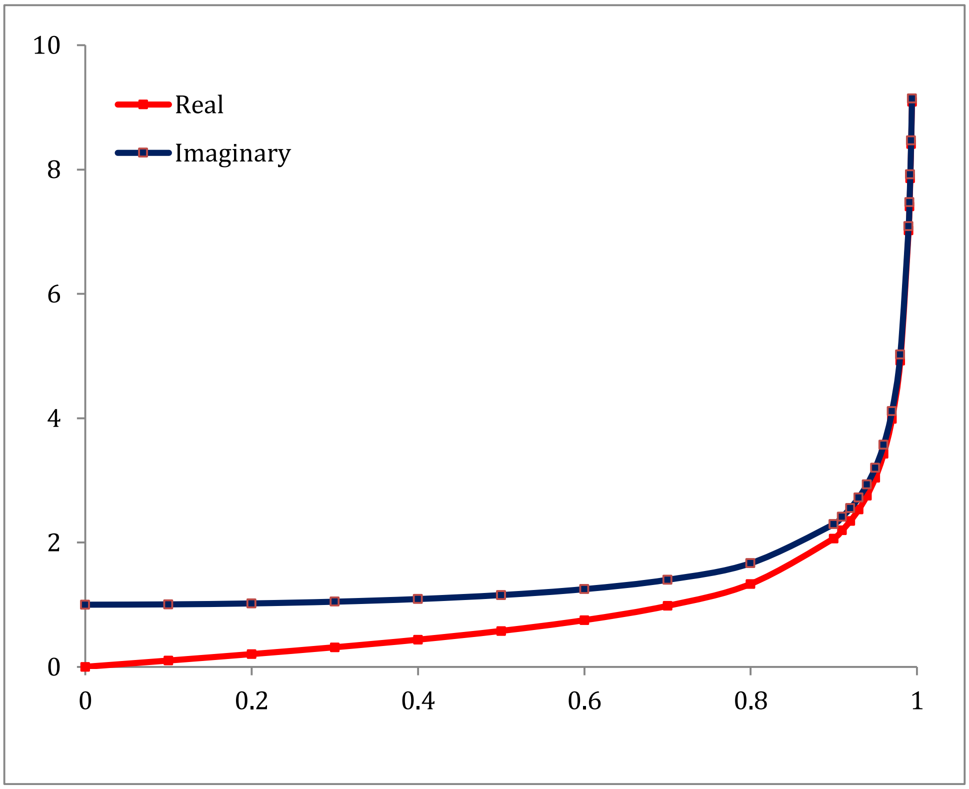

Figure 4: The variation in complex proper velocity's isolated two components

If the relative velocity $u = r/t$ and $u/c = \beta_1$ then physical situation prevails for $\pm\beta$ then, the argument will be law of velocity addition as;

$$Arg(z') = \tan^{-1}\left(\frac{\beta_1 \pm \beta}{1 \pm \beta\beta_1}\right)$$

For $c = 1$, $\beta_1 = u$ and $\pm\beta = \pm v$, then we offer the popular version of Special Relativity;

The resultant velocity will be; $Arg(z') = \frac{u \pm v}{1 \pm uv}$

$\operatorname{Re}\{\upsilon\} = \dfrac{v}{\sqrt{1-v^2}}$ and $\operatorname{Im}\{\upsilon\} = \dfrac{1}{\sqrt{1-v^2}}$ as a function of v (for $c=1$) which clearly reveals that it begins with unit difference at $v=0$ to 0.994 as shown in Figure 4·

Figure 5 depicts the two versions of velocity i.e. complex velocity and proper complex velocity that resembles classical/Newtonian velocity and relativistic velocity respectively.

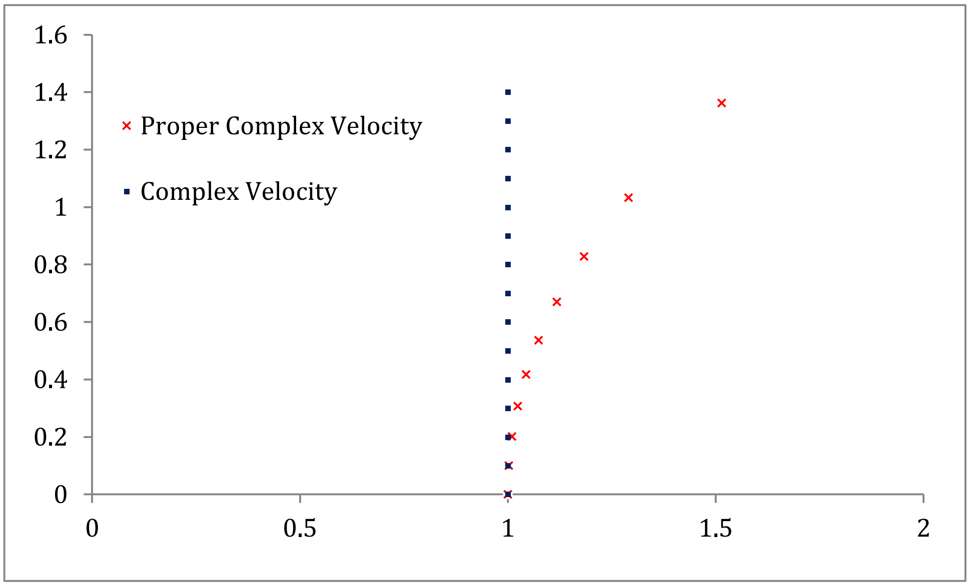

Figure 5: ***Variation of complex velocity*** (ϑ) and ***proper complex velocity*** (υ) with ordinary velocity for scalling $c=1$.

The ***proper complex momentum*** (p) we shall define as a product of proper mass (m_0) and the ***proper complex velocity*** (υ)

$$p = m_0\upsilon$$

$$p = m_0\upsilon = m_0\gamma\frac{dz}{dt} = m_0\gamma(c+iv)$$

Thus the real part $p_x = \gamma\, m_0 c$ and imaginary part $p_y = \gamma\, m_0 v$ reveals that former represents *relativistic momentum* (*relativistic mass* multiplied by *velocity*) later depends on *relativistic mass* or *energy* $\dfrac{E}{c}$.

$$p^2 = \left(P_x^2 - P_y^2\right) + i\left(2P_xP_y\right) = (m_0\gamma)^2\left[\left(c^2 - v^2\right) + i2vc\right] = (m_0c)^2 + i\,2vc(m_0\gamma)^2$$

Here, the quantity $\mathrm{Re}\{p^2\}$ is invariant by definition and possess value $(m_0 c)^2$ that is same for all inertial observers i.e. the observer in another inertial frame finds it $\mathrm{Re}\{p'^2\} = (m_0 c)^2$. Moreover, we know, $E = mc^2 = \gamma m_0 c^2$ hence

$$p_x^2 - p_y^2 = (m_0 c)^2$$

Substituting $p_x = \gamma m_0 c = \frac{E}{c}$ and $p_y = \gamma m_0 v = P$ calling it as relativistic momentum in the last equation, we get;

$$\left(\frac{E}{c}\right)^2 - P^2 = (m_0 c)^2$$

This the famous relativistic energy equation; $E^2 = P^2 c^2 + m_0^2 c^4$. This re-formulates the definition of the proper complex momentum as;

$$p = P + i\frac{E}{c}$$

The ***Lorentz transformation equations for momentum*** with usual notations; $p_x' = \gamma(p_x - \beta p_y)$ and $p_y' = \gamma(p_y - \beta p_x)$. The complete expression for complex representation of momentum can expressed by a single elegant expression as; $p' = p_x' + i\ p_y'$

$$p' = \gamma(p - i\beta\ \bar{p})$$

Here $\bar{p} = p_x - ip_y$ represents complex conjugate of $p = p_x + ip_y$ and the reader may explore $\mathrm{Re}\{p'^2\} = \mathrm{Re}\{p^2\} = -m_0^2 c^2 = S^2$. Its again a hyperbola of similar form and grid as discussed above.

The ***proper complex acceleration*** (α)we shall obtain by differentiating the proper complex velocity with respect to proper time.

$$\alpha = \frac{d\upsilon}{d\tau} = \frac{d}{d\tau}\gamma(v + i\ c) = \frac{d}{d\tau}(v\gamma) + ic\frac{d\gamma}{d\tau} = \gamma\frac{d}{dt}(v\gamma) + ic\gamma\frac{d\gamma}{dt}$$

The real-part: $\gamma(v\dot{\gamma} + \dot{v}\gamma) = a\beta^2\gamma^4 + a\gamma^2 = a\gamma^2(\beta^2\gamma^2 + 1) = a\gamma^4$

The imaginary-part: $c\gamma\dot{\gamma} = a\beta^3\gamma^4$

(Note that, here, $\dot{\gamma} = \frac{1}{v} a\beta^2\gamma^3$, the usual acceleration $a = \dot{v}$ and $\beta^2\gamma^2 + 1 = \gamma^2$)

Thus, we have proper complex acceleration, $\alpha = \frac{d\upsilon}{d\tau} = a\gamma^4\left(1 + i\beta^3\right)$. By the conformal mapping of the proper acceleration we can find out the complex quantity α^2.

$$\alpha^2 = a^2\gamma^8\left[\left(1 - \beta^6\right) + i2\beta^3\right]$$

$\operatorname{Re}\left\{\alpha^2\right\} = a^2\gamma^8\left(1 - \beta^6\right) = a^2 \frac{1 - \beta^6}{\left(1 - \beta^2\right)^4}$, $\operatorname{Im}\left\{\alpha^2\right\} = 2\beta^3 a^2\gamma^8 = \frac{2a^2\beta^3}{\left(1 - \beta^2\right)^4}$. Our conformal transformation based on Lorentz transformation implicates new invariant quantity depicted as above $\operatorname{Re}\left\{\alpha^2\right\}$.

(Here we have $1 - \beta^6 = (\beta + 1)(\beta - 1)\left(\beta^2 + \beta + 1\right)\left(\beta^2 - \beta + 1\right)$, $1 - \beta^2 = (1 - \beta)(1 + \beta)$)

To obtain ***proper complex force*** f, we differentiate the *proper complex momentum* p with respect to proper time τ;

$$\frac{d}{d\tau} p = \frac{d}{d\tau} P + i\frac{d}{d\tau}\frac{E}{c}$$

$$f = \gamma \frac{d}{dt} P + i\gamma \frac{d}{dt}\gamma m_0 c$$

$$f = \gamma F + i m_0 a\beta\gamma^4$$

Here, we have, usual Newtonian $F = \frac{d}{dt} P$, $\frac{d}{dt}\gamma = \dot{\gamma} = \frac{1}{v} a\beta^2\gamma^3$. Also $m_0 a = F$ hence, we get;

$$f = \gamma F\left(1 + i\beta\gamma^3\right)$$

By definition f^2 will be represented as; $f^2 = \gamma^2 F^2\left[\left(1 - \beta^2\gamma^6\right) + i2\beta\gamma^3\right]$ and the $\operatorname{Re}\left\{f^2\right\} = \gamma^2 F^2\left(1 - \beta^2\gamma^6\right)$ represents the invariant quantity. In SW Argand plane the invariant interval for all inertial observers $S^2 = \gamma^2 F^2\left(1 - \beta^2\gamma^6\right)$ i.e. $F = \frac{S}{\gamma\sqrt{\left(1 - \beta^2\gamma^6\right)}}$.

From the theory of complex numbers we have encountered all the ideas encompassed in Einstein's Special Relativity. Moreover, a few new invariant quantity came in the scope of observation and discussions. The subject can be further explored to establish existing facts and hunt for further new issues that were not in scope of usual approach.

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