Pramana–J. Phys. (2017) **123**: ####
DOI 12.3456/s78910-011-012-3

Time is Real and Space is Imaginary

S W ANWANE*

[1] Department of Physics, Shri Shivaji Education Society Amravati's Science College, Congress Nagar, Nagpur-440012 INDIA.

*Corresponding author. E-mail: swanwane2000@yahoo.com

MS received 31 December 2017; revised 11 January 2018; accepted 11 January 2018

Abstract. The space-time warp suggested by Einstein[1] has been molded into imaginary and real-part to raise an event. This weaving offers invariant quantity on squaring the complex number. In complex numbers context the Lagrangian remains same as in the relativistic mechanics but offers complex Hamiltonian. Here we show up that this manifestation offer additional information in every equation.

Keywords. special relativity; complex numbers; proper time; 4-vectors; space contraction; time dilation; simultaneity; Lagrangian; Hamiltonian; Energy

PACS Nos 03.30.+p; 02.30.Fn; 04.20.-q

1. Introduction

Hypotheses that the time is real and space is imaginary offers the complex number representation of space-time event and its square by following equation.

$$\mathbf{z} = ct + i\, r, \mathbf{z}^2 = s^2 + i\, w^2 \tag{1}$$

The $Re\left\{\mathbf{z}^2\right\} = c^2t^2 - r^2 = s^2$ is identified with the `interval` from special relativity while we encounter with a new quantity $Im\left\{\mathbf{z}^2\right\} = 2ctr = w^2$ which is also invariant quantity deferred in representation. For velocity of light being absolute constant [2] and now $c = 1$, $t^2 - r^2 = s^2$ represents hyperbola in t-r plane intersecting real-time axis at s while $2tr = w^2$ also represents a hyperbola which is tilted by 45^o (counterclockwise). This orients the rectangular composition of real-time axis to the one obtained by anti-clockwise rotation by $\pi/4$, that is, a pair of light-like orthogonal lines lines. Mathematically, we represent light-like orthogonal lines as $T : t-r = 0$ and $R : t+r = 0$ giving us $TR = s^2$ which is like $2rt = w^2$. If we retain meanings of the time and the space over counterclockwise $\pi/4$ rotation, we have a important relation; $w = s$. Fig. 1 presents grid-lines formed by $t = 0, \pm1, \pm2, ...$, $r = 0, \pm1, \pm2, ...$, $t^2 - r^2 = 0, \pm1, \pm2, ...$ and $2tr = 0, \pm1, \pm2, ...$ A reference circle is also drawn to indicate that the vertex of the hyperbole is preserved over rotation. It is obvious that here we have additional bunch of information apart from the "`interval`" of Special Relativity. Here, we are at an advantage of having a deferred form of `interval` which is twice the area of square(light-like)/rectangle(space-like or time-like) in s-w argand plane and as expected remains invariant like `interval` under Lorentz Transformation. Thus, the new version of invariant equation opens possibility of offering additional information. Note that here we differ with the idea mentioned by Carbajal wherein "time" is considered as a imaginary quantity[3]. Similarly an offbeat reporting is also found in the literature where speed of light is treated imaginary by Mehran Rezaei[4].

To set-up proof of area invariance in s-w argand plane consider an inertial observer S' moving with velocity v. The complex event $\mathbf{z}$ and its square $\mathbf{z}'^2$ is represented by equation (2).

$$\mathbf{z}' = ct' + i\, r', \mathbf{z}'^2 = s'^2 + i\, w'^2 \tag{2}$$

Fig. 2 depicts the grid-lines for unit intervals on real ($t = 0, \pm1, \pm2, ...$) and imaginary ($r = 0, \pm1, \pm2, ...$) axis for $c = 1$ in frame S. In the same diagram we also plot grid-lines for unit intervals on real ($t' = 0, \pm1, \pm2, ...$) and imaginary ($r' = 0, \pm1, \pm2, ...$) in frame S' exploring Lorentz transformation equations; $ct' = \gamma\,(ct - \beta r)$ and $r' = \gamma\,(r - \beta ct)$. This picture reveals deformation (stretching/squeezing) of diagonals from square grid into rhombus having acute angle θ and obtuse angle $\pi - \theta$

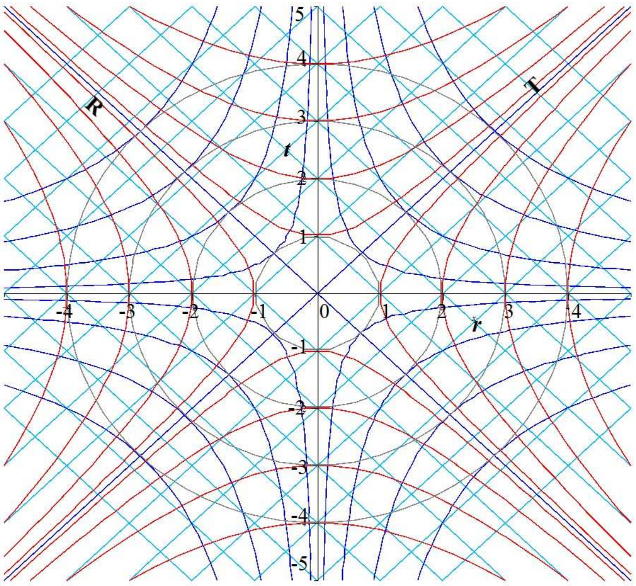

Figure 1. Plot of grid lines formed by $t = 0, \pm1, \pm2, ...,$ $r = 0, \pm1, \pm2, ...,$ $t^2 - r^2 = 0, \pm1, \pm2, ...$ and $2tr = 0, \pm1, \pm2, ...$ along with light-like lines $r = \pm t$, circles of radius showing vertices of all hyperbolas.

as shown in Fig. 2. Here, the speed v of inertial observer S' determines variation in angular compression from $\pi/2$ (between real-imaginary axis in frame S) to the acute angle θ betwen t' and r' in frame S' governed by equation (3).

$$\theta = tan^{-1}\left(\frac{1 - v^2}{2v}\right) \tag{3}$$

The most interesting point here is that the area bound by each unit cells is invariant (obviously unity for observer in frame S and S'). Note that the observer in S finds the stretched argand plane of moving frame showing up grid lines forming rhombus but its area is unity, that is, $t'r' \sin\theta = 1$. We shall identify this in terms of deforming factor $\alpha = \sin\theta$. Note that, the generalized invariance of area is governed by equation $tr \sin\pi/2 = t'r' \sin\theta$.

Thus, we have invariant unit-cell area (deformed shape) in addition to the `interval` from the special theory of relativity is a new bit of information. Note that $\frac{c^2t^2}{s^2} - \frac{r^2}{s^2} = 1$ is an equilateral hyperbola for $c = 1$ with eccentricity $e = \sqrt{2}$ and focus $\left(\pm s\sqrt{2}, 0\right)$ and a pair of directrix $d = \pm s/\sqrt{2}$. The quantity $w = \sqrt{2ctr} = t\sqrt{2} = r\sqrt{2}$ for light-like $(t = r)$ square for $c = 1$ represents diagonal of the square. On the other hand for time-like interval $c^2t^2 > r^2$ the involved velocity v of the object is less than c and is given by $v = r/t$. Therefore, for time-like interval the quantity

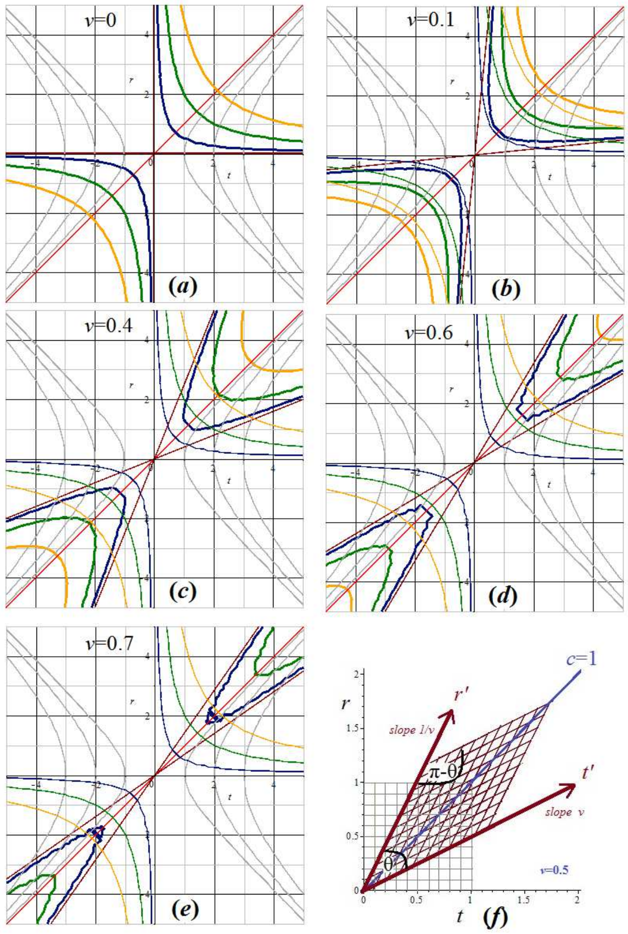

Figure 2. Grid in Frame S governed by lines $t = 0, 1, 2, 3, 4$ and $r = 0, 1, 2, 3, 4$ along with hyperbole $t'^2 - r'^2 = 0, 1, 4, 9$, $2tr = 0, 1, 4, 9$ and $2t'r' = 1, 4, 9$ for (a) $v = 0$ (b) $v = 0.1$ (c) $v = 0.4$ (d) $v = 0.6$ (e) $v = 0.7$ (f) $v = 0.5$; grid lines in frame S and S for $t' = 0, 1$, $t = 0, 1$ and $r' = 0, 1$, $r = 0, 1$ along with overlapping pair of light-lines $r = t$ and $r' = t'$ exhibiting invariant unit are in space-time continuum.

$w = \sqrt{2ctr} = ct\sqrt{2\beta} = t\sqrt{2v}$. From this discussion it is obvious that w grows at rate $\sqrt{2v}$ for time-like/space-like and $\sqrt{2}$ for light-like.

The invariance of interval in two forms thus can be governed by following equation.

$$dw'^2 \sin\theta = dw^2, ds'^2 = ds^2 \tag{4}$$

Fig. 3 presents variation of $\theta(v)$, $\alpha(v)$ and $\frac{d}{dv}\alpha(v)$ with velocity v of the inertial observer over the possible range $0 \le v \le +1$. Here, we have mathematical versions as equation (3) and (5).

$$\alpha(v) = \sin\theta = \frac{1 - v^2}{v\sqrt{4 + \frac{\left(1-v^2\right)^2}{v^2}}}, \frac{d}{dv}\alpha(v) = -\frac{4v}{1 + v^2} \tag{5}$$

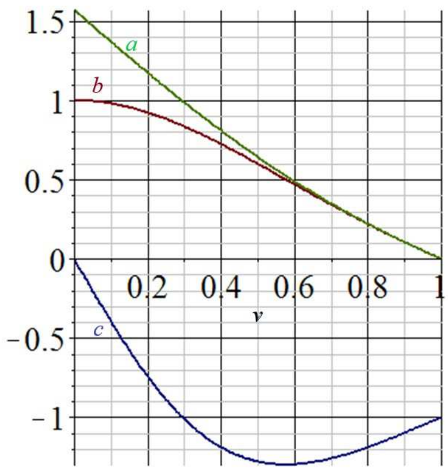

Figure 3. Variation of (i) θ (ii) $\alpha = \sin\theta$ and (iii) $d\alpha/dv$ with velocity under scaling constrain $c = 1$.

Curve analysis in MAPLE 18 for function $\alpha(v)$ offers information as following: (i) the factor α posses local minima at $v = 1$, (ii) there are no increasing interval, (iii) function is decreasing over entire interval $[0, 1]$ (iv) the function is concave down over interval $[0, 0.577]$, and concave up over interval $[0.577, 1]$ (v) the inflation point is interval $[0.577, 0.502]$ at which $\gamma = 1.22437$.

Similar analysis for the function $\frac{d}{dv}\alpha(v)$: (i) it possess local minima at $[0.577, -1.299037747]$, (ii) it is decreasing over interval $[0, 0.577]$ and increasing thereafter over interval $[0.577, 1]$ (iv) the function is concave up over interval $[0, 1]$ no inflation point. However, the variation of θ following equation (3) indicates that θ contentiously decreases with velocity over interval $[0, 1]$, without inflation. The inflation occurring reflects in $\frac{d\theta}{dv}$ also at $v = 0.577$.

To define proper time, we shall fix clock to the origin ($r' = 0$) of frame S' to get rid of the term $Im\left\{\mathbf{z}'^2\right\}$. This attempt formulates legacy to define proper time. Using notation $\tau = t'$ for the "proper time" and $\beta = dr/cdt = v/c$ becomes following;

$$\frac{dt}{d\tau} = \frac{1}{\sqrt{1-\beta^2}} = \gamma \tag{6}$$

As the quantity proper time beholds universality that all inertial observers agree upon, we shall differentiate the complex event w.r.t. τ and explore chain rule to relate with our usual parameters like velocity, momentum and acceleration.

Moreover, the term α can also be expressed as;

$$\alpha = \sin\theta = \left(\frac{dw}{dw'}\right)^2 = \frac{drdt}{dr'dt'} = \gamma\frac{dr}{dr'} \tag{7}$$

2. Proper complex velocity v

is defined as the proper time rate of change of the complex position $\mathbf{z}$. Mathematically, $\mathbf{v} = d\mathbf{z}/d\tau = \breve{\mathbf{z}}$ and $d\mathbf{z}/dt = \dot{\mathbf{z}}$ and chain rule offers; $\breve{\mathbf{z}} = \gamma\dot{\mathbf{z}}$

$$\mathbf{v} = \breve{\mathbf{z}} = \gamma\dot{\mathbf{z}} = \gamma\,(c + i\,\dot{r}) \tag{8}$$

Here, $\dot{r} = u$ is usual velocity in frame S. The square conformal[5] mapping of complex proper velocity will be;

$$\mathbf{v}^2 = \gamma^2\left(c^2 - \dot{r}^2 + i\,2c\dot{r}\right) \tag{9}$$

In case the velocity involved $\dot{r} = u = v$ i.e. the object is stationary in frame S', the above equation reduces to;

$$\mathbf{v}^2 = c^2 + i\,2cv\gamma^2 \tag{10}$$

3. The Lorentz Transformation using Complex Number

The Lorentz Transformation[6] (LT) equations with all usual meanings are; $ct' = \gamma\,(ct - \beta r)$ and $r' = \gamma\,(r - \beta ct)$ now can be expressed by the following unique equation.

$$\mathbf{z}' = \gamma\,(\mathbf{z} - i\,\beta\overline{\mathbf{z}}) \tag{11}$$

here, $\overline{\mathbf{z}}$ is complex conjugate[5] of $\mathbf{z}$ while $\mathbf{z}'$ represents the complex number representation of an event in another inertial frame S' moving with velocity v w.r.t. frame S. Moreover, the inverse Lorentz Transformation can be expressed as; $\mathbf{z} = \gamma\left(\mathbf{z}' + i\,\beta\overline{\mathbf{z}}'\right)$

4. Velocity Composition in s-w argand plane

An event chart in frame S and S' can be expressed as depicted in equation (1) and (2). Differentiating equations (1) and (2) with respect to t and τ (exploring chain rule using equation (6)) we get velocity in respective frames in complex number form. The complex velocity and proper complex velocity can be expressed as; $\dot{\mathbf{z}} = c + i\,\dot{r}$,$\breve{\mathbf{z}} = \gamma\,(c + i\,\dot{r})$

Under inverse Lorentz Transformation, the event $d\mathbf{z}$ from equation (11) divided by $dt = \gamma d\tau$ offers velocity relation between two frames.

$$\frac{d\mathbf{z}}{dt} = \gamma^2 \left(\frac{d\mathbf{z}'}{d\tau} + i\,\beta \frac{d\overline{\mathbf{z}}'}{d\tau} \right) \tag{12}$$

Simplifying above equation and using our shorthand notations/equations $u = \dot{r}, u' = dr'/d\tau, \gamma = \left(1 - \beta^2\right)^{-0.5}$ and comparing real & imaginary parts we get a pair of relations;

$$\gamma^2 = \frac{c}{c + \beta u}, u' = \gamma^2 (u + v) \tag{13}$$

Combining above pair of equations, we get the same expression for velocity composition as in special relativity;

$$u' = \frac{u + v}{1 + \frac{uv}{c^2}} \tag{14}$$

Thus, we smoothly validates velocity composition rule of special relativity.

4.1 *Proper complex velocities* $\mathbf{v}, \mathbf{v}'$

and the Lorentz Transformation will enable us with a new bunch of information to explore. We have $\check{\mathbf{z}} = \gamma (c + i\,\dot{r})$ and $\check{\mathbf{z}}' = \gamma \frac{d}{dt} (\mathbf{z} - i\,\beta \overline{\mathbf{z}})$ representing the velocity. The square conformal mapping of this pair is invariant for $(\dot{r} = v, \check{r}' = v')$ as;

$$c^2 - v'^2 = \gamma^2 \left(c^2 - v^2\right), \gamma^2 v = v' \sin\theta \tag{15}$$

In the above pair of equations, the first equation represents $v' = 0$ which means that a stationary particle in S' and its velocity is v in S. However, the later equation in additional information emerging out where the angle $\theta = \sin^{-1}\left(\gamma^2 \frac{v}{v'}\right)$ is angle subtended by t' and r' axis which resembles equation (7). The Lorentz transformation equation for the complex proper velocity will be;

$$\check{\mathbf{z}}' = \gamma \left(\dot{\mathbf{z}} - i\,\beta \dot{\overline{\mathbf{z}}}\right), \check{\mathbf{z}} = \gamma \left(\dot{\mathbf{z}}' + i\,\beta \dot{\overline{\mathbf{z}}}'\right) \tag{16}$$

4.2 *Proper complex momentum* $\mathbf{p}$

is defined as the product of proper mass m_o and proper complex velocity $\mathbf{v}$. Mathematically, $\mathbf{p} = m_o \mathbf{v} = m_o \check{\mathbf{z}}$.

$$\mathbf{p} = m_o \gamma (c + i\,\dot{r}) \tag{17}$$

Using pre-hand information; $mc = E/c$ along with relativistic momentum $p = m\dot{r}$ for $m = m_o \gamma$;

$$\mathbf{p} = \frac{E}{c} + i\,p \tag{18}$$

The square conformal mapping of complex proper momentum using equation (20) and (21) we get;

$$\mathbf{p}^2 = m_o^2 c^2 + i\,2m^2 vc = \frac{E^2}{c^2} - p^2 + i\,2\frac{Ep}{c} \tag{19}$$

Comparing the real and imaginary part, we get a pair of equations;

$$E = \sqrt{m_o^2 c^4 + p^2 c^2}, E = mc^2 \tag{20}$$

Thus, we have two famous relativistic equations stating energy of the particle. Moreover, the Lorentz transformation equation for complex proper momentum will be;

$$\mathbf{p}' = \gamma (\mathbf{p} - i\,\beta \overline{\mathbf{p}}), \mathbf{p} = \gamma \left(\mathbf{p}' + i\,\beta \overline{\mathbf{p}}'\right) \tag{21}$$

4.3 *Proper complex acceleration* $\mathbf{a}$

is defined as the proper time rate of change of the complex velocity $\mathbf{v}$. Mathematically, $\mathbf{a} = d\mathbf{v}/d\tau$ along with usual notation $a = \dot{v}$ and inter-relations $\dot{\gamma} = \frac{va\gamma^3}{c^2}$, $1 + \beta^2 = \gamma^2$ we have;

$$\mathbf{a} = \gamma \frac{d}{dt} \gamma (c + iv) = c\gamma\dot{\gamma} + i\left(\gamma^2 \dot{v} + \gamma\dot{\gamma}v\right) = a\gamma^4 (\beta + i) \tag{22}$$

The square conformal mapping of complex proper acceleration will be;

$$\mathbf{a}^2 = -a^2\gamma^6 + i\,2\beta a^2 \gamma^8 \tag{23}$$

The Lorentz transformation for proper complex acceleration can be obtained by dividing equation (18) for velocity transformation by $d\tau = \gamma^{-1} dt$ as;

$$\mathbf{a}' = \gamma^3 \left(\mathbf{a} - i\,\beta \dot{\overline{\mathbf{a}}}\right) \tag{24}$$

5. The Lagrangian L

can be found from the action integral in reference to the complex number representation of the event as;

$$S = -\alpha \int_a^b dz = -\alpha \int_a^b \sqrt{ds^2 + i\, dw^2} \tag{25}$$

Here, we have pair of complex invariant intervals as; $d\mathbf{z}^2 = c^2dt^2 - dr^2 + i\ 2cdrdt \sin\pi/2$ and similarly, $d\mathbf{z}'^2 = c^2dt'^2 - dr'^2 + i\ 2cdr'dt' \sin\theta$ obeying invariance governed by; $d\mathbf{z}'^2 = d\mathbf{z}^2$. Fixing moving object to the origin of frame S' we have $dr' = 0$, reduces invariant pairs of equations to $ds = cdt' = c\gamma^{-1}dt$ and $dw = 0$ which simplifies the action to the traditional one in the relativistic mechanics.

$$S = -\int_a^b \alpha c \gamma^{-1} dt \tag{26}$$

Consequently the Lagrangian of the particle will also be unaltered.

$$\mathbf{L} = -\alpha c \sqrt{1 - \frac{v^2}{c^2}} \tag{27}$$

It obviously follows from the relativistic mechanics[6] that α characterizes particle and the limiting conditions formulates $\alpha = m_o c$ for free particle. Thus, our Lagrangian (being free from imaginary-part) also resembles completely with the traditional.

$$\mathbf{L} = -m_o c^2 \sqrt{1 - \frac{v^2}{c^2}} \tag{28}$$

6. Energy and momentum

Let us workout $\partial\mathbf{L}/\partial\mathbf{v}$ for the Lagrangian and proper complex velocity from equation (30) and (27) respectively.

$$\frac{\partial\mathbf{L}}{\partial\mathbf{v}} = \frac{\partial\mathbf{L}}{\partial v}\frac{\partial v}{\partial\mathbf{v}} \tag{29}$$

The quantity $\partial\mathbf{L}/\partial v = m_o\gamma v = mv = p_o\gamma$ is relativistic momentum. However, the quantity $\partial v/\partial\mathbf{v}$ needs to be evaluated and analyzed. For convenience, let us first find out $\partial\mathbf{v}/\partial v$ instead.

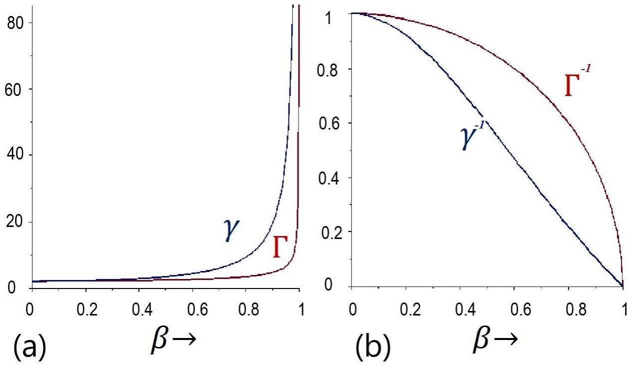

Figure 4. Variation of (a) Γ and γ (b) $1/\Gamma$ and $1/\gamma$ with β, i.e. velocity under scaling constrain $c = 1$.

$$\frac{\partial\mathbf{v}}{\partial v} = \frac{\partial}{\partial v}\frac{c + i\,v}{\sqrt{1 - \frac{v^2}{c^2}}} = \beta\gamma^3 + i\,\gamma\left(1 + \beta^2\gamma^2\right) \tag{30}$$

$$\frac{\partial\mathbf{v}}{\partial v} = \gamma^3(\beta + i) \Rightarrow \frac{\partial v}{\partial\mathbf{v}} = \frac{(\beta - i)}{\gamma^3(1 + \beta^2)} \tag{31}$$

Thus,

$$\frac{\partial\mathbf{L}}{\partial\mathbf{v}} = p_o\Gamma^{-1}(\beta - i) \tag{32}$$

for shorthand notation $\Gamma = \frac{1+\beta^2}{1-\beta^2}$. Moreover, it is worth mentioning that the γ and Γ almost behave similar. The Fig. 4a depicts that the γ follows Γ over entire range of β. The dis-contentious functions can be looked in an interesting way as γ^{-1} and Γ^{-1} which is also depicted in the Fig. 4b .

It is worth noting that the proper complex momentum is very close to the momentum defined from the mechanics by differentiating the Lagrangian with respect to the proper complex velocity[6]. Thus, here we have a very important finding of the present work as $\mathbf{p} \approx \frac{\partial\mathbf{L}}{\partial\mathbf{v}}$.

7. The Hamiltonian

The quantity $\mathbf{pv} - \mathbf{L}$ forms energy equation which also offers complex Hamiltonian of the form;

$$\mathbf{E} = \mathbf{pv} - \mathbf{L} = m_o c^2\left[\left(1 - \gamma^{-1}\right) + i\ 2\gamma^2\beta\right] = \mathbf{H} \tag{33}$$

The $Re\{\mathbf{E}\} = \frac{1}{2}m_o v^2$ represents Newtonian kinetic energy under limiting condition $\beta \to 0$. While $Im\{\mathbf{E}\} =$

$2m_o c^2 \beta\gamma^2$ vanishes under limiting condition $\beta \to 0$. This offers smooth return to the Newtonian theory for low speeds for which we have started from the action is $S = -m_o c \int_a^b dz$ and the Principle of Least action $\delta S = 0$.

8. Discussion and Conclusion

The time-space warp suggested by Einstein in 1905 has been put forth in present work as a real part and imaginary part to form complex number representation of an event. ***Special Relativity*** suggests that space-time `interval` of the format $s = \sqrt{t^2 - r^2}$ remains invariant unlike $\sqrt{t^2 + r^2}$ in the circular geometry, over rotation of coordinate axis about z-axis. Now, the real part of the square conformal mapping of an event offers invariant interval like in Special Relativity. However, this work also offers a finding that the imaginary part too sublimes an invariant quantity in deferred form. This new quantity is attributed to invariant area in space-time continuum of argand plane over Lorentz Transformation. The idea of exploring complex number representation of an event in special relativity with the *hypothesis* that ***time is real and space is imaginary*** is found to progress smoothly like 4-vectors[6] treatment and efficiently offers an additional novel information from its imaginary counterpart. One of the important findings is inflation of the rate at which the deforming factor α varies with the relative velocity v at $v = 0.577c$. Moreover, the proposal of hypothesis that ***time is real and space is imaginary*** is supported as the new quantity $\mathbf{z}^2$ is invariant over *Lorentz Transformation*. Nevertheless, both entities ($Re\left\{z^2\right\}, Im\left\{z^2\right\}$) are hyperbole of the same family except rotation (by $\pi/4$) and squeezing by θ governed by the relative velocity involved. This quantity encompasses the Einstein's contention of space-time `interval` while rendering an additional invariant quantity that possess dimensions of space-time area. The established Lorentz Transformation equations are replaced by a single elegant equation and the other counter parts like; *velocity composition*; *time dilation*; *space contraction* and *relativistic Lagrangian* etc are encompassed without loss. Moreover, the present work proposes new ***Hamiltonian*** in complex form. One of the important features of the present work is that; the proper complex momentum is very close to the momentum defined from the mechanics obtained by differentiating the Lagrangian with respect to the proper complex velocity and the new relativistic complex Hamiltonian smoothly rolls back to the classical Hamiltonian representing kinetic energy.

References

[1] A Einstein *Ann d Phys* **17** 891 (1905)
[2] A A Michelson, E H Morley *Am J Sci* **34** 333 (1887)
[3] A Carbajal *Nature and Science* **4(1)** 23 (2006)
[4] M Rezaei *WSEAS Trans on Comm IRAN* ISSN: 1109-2742 **2(9)** 120 (2010)
[5] Ervin Kreyszig *Advanced Engineering Mathematics Wiley-India Ohio* **10** (2010)
[6] L Landau, E Lifshitz *The Classical Theory of Fields -translated by Morton Hamermesh Addison-Wesley* **2** 100 (1951)